UNIT - 4

Random Processes - Temporal characteristics

Random process is defined as a collection of functions of time with a probability measure (or) Random variable that is a function of time is called a random process.

→ A random process is a family of functions $\{X = x(s,t)\}$, $s \in S$ and $t \in T\}$, defined with random variables $\{X = x\}$ on sample space S with a function of time "T". Thus a random process is a function of two variables, sample space and time.

→ It is denoted as $x(s,t)$ or $x(t)$. In the random process $x(s,t)$,

→ s is made fixed, the random process is a function of time only ie $x(s,t)$ is a time function

→ "t" is made fixed, the random process is a function of s-only ie $x(t,s)$ is a random variable.

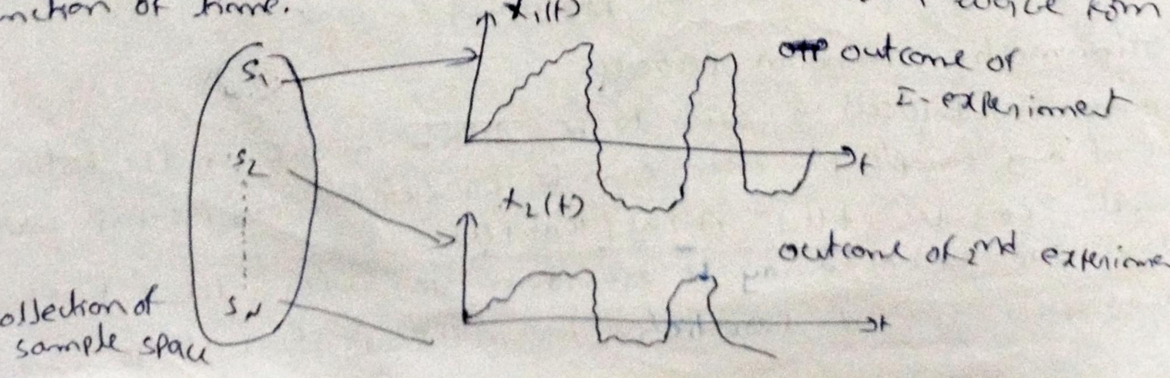
→ s and t are both fixed then $x(s,t)$ is a number.

Example:: Let R.V "x" is a temperature, let us consider we want to record the temperature on a certain room. If we take different values and different days. its value should be recorded for many no. of days, since temperature varies from morning to evening. The values are to be recorded for different times on one day and also for many days

Thus the above R.V x is a function of time can be expressed as $x(t)$. which is known as random process.

→ Difference between R.V and Random Process::

For R.V the out come of experiment is mapped into a number. For random process the outcome is mapped into a wave form ie a function of time.



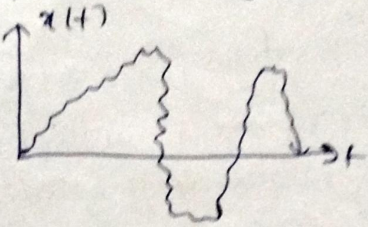
classification:

The random process is classified into 4 types based on time "t" & R.V "x"

1) continuous random process:

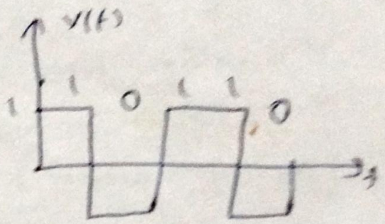
If the R.V "x" is continuous and time "t" is continuous then the random process is called continuous random process.

ex: The temperature measured w.r.t time on a day.



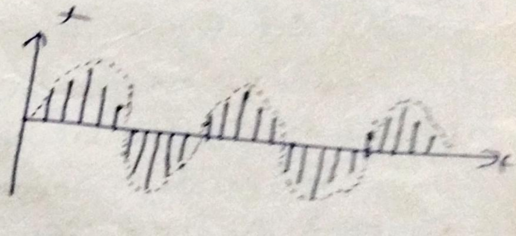
2. Discreate Random process

If R.V "x" is a discrete and time "t" is continuous then x(t) is referred to as discrete random process.



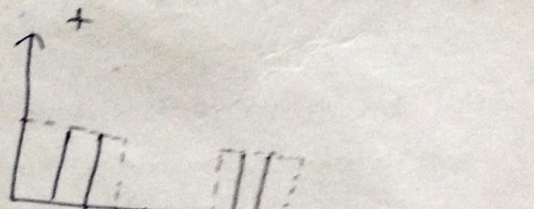
EX:- A digital encoded signal has only two discrete values a +ve level and a -ve level but the time is continuous

3. continuous random sequence:



If r.v x is a continuous and time has discrete values then r.v is referred to as continuous random sequence. It is defined only discrete time instants.

4. Discrete random sequence:



If r.v "x" is discrete and time "t" is discrete then the random process is called discrete random sequence.

Discrete random sequence can be obtained by sampling and quantizing. This process is known as digital process. It is used in digital signal processing

deterministic random process:

A random process is said to be deterministic if the future values of any sample function can be predicted from its past values.

Example consider $x(t) = A\cos(\omega_0 t + \theta)$.

Here A, ω_0, (or) θ may be random variables. the future values of any r.v can be predicted from its past values.

Non Deterministic Random Process

if future values of any sample function can not be predicted from observed past values. The process is called non-deterministic random process.

Stationary random process: A Random process is said to be stationary if all its statistical properties (mean, variance, joint moments---) do not change w.r.t time is called stationary random process otherwise it is non-stationary.

Distribution & Density functions of a Random Process

Distribution function of R.P

Let us consider a random process $X(t)$

For a single random variable at time "t_1" ie $X_1 = X(t_1)$, the distribution function is defined as

$$F_X(x_1 : t_1) = P\{X_1(t_1) \leq x_1\} \quad -(1)$$ where x_1 is any real number.

The function $F(x_1 : t_1)$ is called first order distribution function of R.P.

→ For two random variable ie at times t_1 & t_2 ie $X_1 = X(t_1)$, $X_2 = X(t_2)$, the 2nd order joint distribution function is given by

$$F_X(x_1 x_2 : t_1 t_2) = P\{X(t_1) \leq x_1, X(t_2) \leq x_2\} \quad -(2)$$

similarly for n-random variables ie $X_i = X(t_i)$ $i = 1, 2, 3 --- N$ the nth order joint distribution function is given by

$$F_X(x_1 x_2 --- x_N ; t_1 t_2 --- t_N) = P\{X(t_1) \leq x_1, X(t_2) \leq x_2 --- X(t_N) \leq x_N\} \quad -(3)$$

Density function of Random Process :-

The derivative of the first order distribution function w.r.t x_1 is called 1st order joint density function

$$f_X(x_1 : t_1) = \frac{d}{dx_1} F_X(x_1 : t_1) \quad -(4)$$

second order joint density function

$$f_X(x_1 x_2 ; t_1 t_2) = \frac{\partial^2 F_X(x_1, x_2 ; t_1 t_2)}{\partial x_1 \, \partial x_2} \quad -(5)$$

similarly nth order joint density function

$$f_X(x_1 x_2 --- x_N ; t_1 t_2 --- t_N) = \frac{\partial^N F_X(x_1, x_2 --- x_N ; t_1 t_2 --- t_N)}{\partial x_1 \, \partial x_2 --- \partial x_N} \quad -(6)$$

Statistically independent random process :-

Consider two random processes $x(t)$ and $y(t)$ let $x(t)$ has random variables $x(t_1), x(t_2) \dots x(t_N)$ with joint density function $f_x(x_1, x_2 \dots x_N; t_1 t_2 \dots t_N)$ and $y(t)$ has random variables $y(t_1'), y(t_2') \dots y(t_n')$ with joint density function $f_y(y_1, y_2 \dots y_n; t_1' t_2' \dots t_n')$, the two random processes $x(t)$ and $y(t)$ are said to be statistically independent if the random variable group $x(t_1), x(t_2), \dots x(t_n)$ is independent of group $y(t_1')\ y(t_2') \dots y(t_n')$ for any choice of times $t_1 t_2 \dots t_n$, $t_1' t_2' \dots t_n'$. ie the joint density function of two Random processes $x(t)$ and $y(t)$ is equal to the product of the individual joint density function of $x(t)$ and $y(t)$. Mathematically

~~$f_{xy}(x_1, x_2, \dots x_N; t_1, t_2 \dots t_N)$~~

$$f_{xy}(x_1, x_2 \dots x_N, y_1 y_2 \dots y_n; t_1 t_2 \dots t_N; t_1' t_2' \dots t_n') =$$
$$f_x(x_1, x_2 \dots x_N; t_1 t_2 \dots t_N) \cdot f_y(y_1 y_2 \dots y_n; t_1' t_2' \dots t_n').$$

First-order stationary Process :-

A Random Process is said to be stationary to order 1 (or) first order stationary, if its first order density function does not change with time or shift in the time value.

ie $f_x(x_1; t_1) = f_x(x_1; t_1 + \Delta)$ — ①

For any t_1 and any real number Δ.

since equation ①, $f_x(x_1; t_1)$ is independent of t_1. the mean value of this process is constant. Therefore the condition for 1st order - stationary Random processes its mean value is constant ie

$$E[x(t)] = \bar{x} = \text{constant}$$

proof :- let us consider a random process $x(t)$ with R.vs $x_1 = x(t_1)$ and $x_2 = x(t_2)$ at time instants t_1 & t_2 equals to $t_1 + \Delta$.

$$E[x(t_1)] = \int_{-\infty}^{\infty} x_1 f_x(x_1; t_1)\, dx_1 \text{ — ①}$$

$$E\{x(t_2)\} = \int_{-\infty}^{\infty} x_2 f_x(x_2; t_2)\, dx_2 \text{ — ②}$$

put $t_2 = t_1 + \Delta$.

$$E[X(t_1+\Delta)] = \int_{-\infty}^{\infty} x_1 f_X(x_1; t_1+\Delta)\, dx_1$$

For stationary process

$$f_X(x_1; t_1) = f_X(x_1; t_1+\Delta)$$

$$E[X(t_1+\Delta)] = \int_{-\infty}^{\infty} x_1 f_X(x_1; t_1)\, dx$$

$$= E[X(t)] = \bar{X} = \text{constant}$$

Second order and wide sense stationary (WSS) process:-

A R.P is said to be stationary to order 2 or second order stationary process, if its second order joint density function do not change with time (or) second order density function satisfies

$$f_X(x_1, x_2; t_1, t_2) = f_X(x_1, x_2; t_1+\Delta, t_2+\Delta) \quad \text{— (1)} \quad \text{for any } t_1, t_2 \text{ \& } \Delta.$$

Now the correlation $E[X_1 X_2] = E[X(t_1) X(t_2)]$ of a random process be a function of t_1 and t_2. This function is denoted by $R_{XX}(t_1, t_2)$.

The auto correlation of the Random process $X(t)$ is given by

$$R_{XX}(t_1 t_2) = E[X(t_1) . X(t_2)] \quad \text{— (2)}$$

→ The condition for a 2nd order stationary process is its auto correlation function depends only on time differences and not on absolute time. ie

if X_1 and X_2 are two R.Vs of a R.P $X(t)$ defined at t_1 and t_2. Then the auto correlation function is

$$R_{XX}(t_1 t_2) = E[X(t_1) X(t_2)]$$

if $\tau = t_2 - t_1$

then $R_{XX}(t_1; t_1+\tau) = E[X(t_1) X(t_1+\tau)] = R_{XX}(\tau)$ — (3)

Wide sense stationary process:

If a Random process $X(t)$ is a second order stationary process then it is called wide sense stationary process or weak sense stationary process "X". The conditions for WSS process are

(1) $E[X(t)] = \bar{X} = \text{constant}$

(2) $R_{XX}(t, t+\tau) = E[X(t) X(t+\tau)] = R_{XX}(\tau)$ is independent of absolute time "t".

nth order strict sense stationary (SSS) process :-

A random process $x(t)$ is said to be strict sense stationary if its nth order joint density function does not change with time ie

$$f_x(x_1, x_2 \cdots x_N; t_1 t_2 \cdots t_N) = f_x(x_1, x_2 \cdots x_N; t_1+\Delta, t_2+\Delta, \cdots t_N+\Delta)$$

for any $t_1, t_2 \cdots t_N$ and Δ.

Problem:- Determine whether the Random process $x(t) = A\cos(\omega_0 t+\theta)$ is wide sense stationary or not? Here A, ω_0 are constants & θ is uniformly distributed over an interval $(0, 2\pi)$.

sol:-

Given here Random process $x(t) = A\cos(\omega_0 t+\theta)$ and A, ω_0 are constants and 'θ' is uniformly distributed R.V over an interval $(0, 2\pi)$.

$$f_\theta(\theta) = \begin{cases} \frac{1}{2\pi} & 0 < \theta < 2\pi \\ 0 & \text{else} \end{cases}$$

$$E[x(t)] = \int_{-\infty}^{\infty} (A\cos(\omega_0 t+\theta)) f_\theta(\theta)\, d\theta = \int_0^{2\pi} A\cos(\omega_0 t+\theta) \cdot \frac{1}{2\pi} d\theta$$

$$= \frac{A}{2\pi} \cdot \frac{\sin(\omega_0 t+\theta}{1} \Big|_0^{2\pi} = \frac{A}{2\pi}\left[\sin(\omega_0 t+2\pi) - \sin\omega_0 t\right].$$

$$= \frac{A}{2\pi}\left[\sin\omega_0 t - \sin\omega_0 t\right] = 0 = \text{constant}$$

consider auto correlation $R_{xx}(t, t+T) = E([x(t)\, x(t+T)])$

$$= E[A\cos(\omega_0 t+\theta) . A\cos(\omega_0(t+T)+\theta)]$$

$$= A^2 E[\cos(\omega_0 t+\theta)\cos(\omega_0 t+\omega_0 T+\theta)].$$

$$= \frac{A^2}{2} E[\cos(2\omega_0 t+\omega_0 T+2\theta) + \cos(-\omega_0 T)].$$

$$= \frac{A^2}{2}\left[E[\cos(2\omega_0 t+\omega_0 T+2\theta)] + E[\cos(\omega_0 T)]\right].$$

$$= \frac{A^2}{2}\left\{\int_0^{2\pi} \cos(2\omega_0 t+\omega_0 T+2\theta) \cdot \frac{1}{2\pi} d\theta + \int_0^{2\pi} \cos(\omega_0 T) \frac{1}{2\pi} d\theta\right\}$$

$$= \frac{A^2}{2}\left\{\sin(2\omega_0 t+\omega_0 T+2\theta) \cdot \frac{1}{2\pi} \cdot \frac{?}{?}\Big|_0^{2\pi} + \cos\omega_0 T\right\}$$

$$= \frac{A^2}{2}\left\{\underbrace{\sin(2\omega_0 t+\omega_0 T+4\pi)}_{0} \cdot \frac{1}{2\pi} + \cos\omega_0 T\right\}$$

$$= \frac{A^2}{2}\cos\omega_0 T.$$

Auto correlation is independent of time "t" & function of τ.
∴ Mean of the R.P is constant i.e $E[x(t)] = 0$ & the auto correlation function is independent of absolute time "t" hence given Random process is WSS.

Statistical or ensemble average:-

Consider a random process $x(t)$, the two statistical averages are mostly used in random process are mean and auto correlation function.

→ The mean value of a random process $x(t)$ is the expected value of $x(t) = \bar{x} = E[x(t)]$.

→ The auto correlation of a random process $x(t)$ is denoted as $R_{xx}(t, t+\tau) = E[x(t)\, x(t+\tau)]$.

Time average:-

The time average of a quantity is defined as

$$A[\cdot] = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} [\cdot]\, dt \rightarrow (1)$$

Here A is used to denote the time average which is similar to E for statistical average.

→ The time average mean of a sample function $x(t)$ of a random process $X(t)$ is defined as

$$A[x(t)] = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} x(t)\, dt \quad -(2)$$

→ The time average auto correlation of a sample function $x(t)$ is defined as

$$\bar{R}_{xx}(t, t+\tau) = A[x(t)\, x(t+\tau)] = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} x(t)\, x(t+\tau)\, dt \quad -(3)$$

Ergodic Random process:-

A Random process $X(t)$ is said to be Ergodic random process if all the time averages of a sample function $x(t)$ of a random process $X(t)$ equals to the corresponding statistical (or) Ensemble averages of $X(t)$

Mean Ergodic Process :-

A random process $X(t)$ is said to be mean Ergodic (or) Ergodic in mean if the statistical average of mean $\bar{X}$ equals to the time average $\bar{x}$ of any sample function $x(t)$ with probability one for all sample functions ie

$$A[x(t)] = \bar{x} \quad , \quad E[X(t)] = \bar{X}$$

$$\Rightarrow E[X(t)] = \bar{X} = A[x(t)] = \bar{x}$$

Auto correlation Ergodic Process :-

In a stationary random process $X(t)$ is said to be auto correlation ergodic (or) Ergodic in Auto correlation if and only iff its time auto correlation function of any sample function $x(t)$ is equals to the ~~st~~ statistical Auto correlation function of $X(t)$

ie $E[X(t)X(t+\tau)] = A[x(t)x(t+\tau)]$ (or)

$$R_{XX}(t, t+\tau) = \bar{R}_{XX}(t, t+\tau).$$

Problem:- A R.P is described by $X(t) = A^2 \cos^2(\omega_c t + \theta)$ where A and ω_c are constants and θ is uniformly distributed b/w $(-\pi, \pi)$. Is $X(t)$ WSS stationary or not.

→ consider a R.P $X(t)$ is given by $X(t) = A\cos(\omega t + \theta)$ where ω and θ constants and A is a Random variable, determine wheather $X(t)$ is WSS or not

Sol: Given $X(t) = A\cos(\omega t + \theta)$

Here ω and θ are constants and A is a R.V.

$E\{X(t)\} = E[A\cos(\omega t + \theta)] = \cos \omega t + \theta \; E[A]$

Here $E\{X(t)\}$ is constant only if $E[A] = 0$

$R_{XX}(t, t+\tau) = E\{X(t)X(t+\tau)\} = E[A\cos(\omega t+\theta) A\cos(\omega(t+\tau)+\theta)]$

$= \cos(\omega t+\theta)\cos(\omega t + \omega\tau + \theta) E(A^2)$

$= \frac{1}{2}[\cos(2\omega t + \omega\tau + \theta) + \cos\omega\tau] E(A^2)$

In this auto correlation is a function of "t" hence given R.P is not a WSS

→ Given the random process $x(t) = A\cos w_0 t + B\sin w_0 t$ where w_0 is a constant and A and B are uncorrelated zero mean R.Vs having different density functions but the same variance σ^2. Show that $x(t)$ is wide sense stationary but not strict sense stationary.

Soly:: Given that $x(t) = A\cos w_0 t + B\sin w_0 t$

Here w_0 is constant and A, B are uncorrelated zero mean random variables. i.e $E(A) = 0$ and $E(B) = 0$

Given A & B are uncorrelated $E[AB] = E[A]E[B] = 0$.

and variance of $A = E(A^2) - [E(A)]^2 = \sigma^2$

$E(A^2) = \sigma^2$ $\quad [\because E[A] = 0$

similarly $E[B^2] = \sigma^2$

consider $E[x(t)] = E[A\cos w_0 t + B\sin w_0 t]$

$= \cos w_0 t\, E[A] + \sin w_0 t\, E[B] = \cos w_0 t(0) + \sin w_0 t(0)$
$= 0$.

consider auto correlation function.

$R_{xx}(t, t+\tau) = E[x(t)\, x(t+\tau)]$

$= E[(A\cos w_0 t + B\sin w_0 t)(A\cos(w_0(t+\tau)) + B\sin(w_0(t+\tau)))]$

$= E[A^2\cos w_0 t\cos w_0(t+\tau) + AB\cos w_0 t\sin w_0(t+\tau) +$
$AB\sin w_0 t\cos w_0(t+\tau) + B^2\sin w_0 t\sin w_0(t+\tau)]$

$= \cos w_0 t\cos w_0(t+\tau)E[A^2] + \cos w_0 t\sin w_0(t+\tau)E[AB] +$
$\sin w_0 t\cos w_0(t+\tau)E[AB] + \sin w_0 t\sin w_0(t+\tau)E[B^2]$

$= \cos w_0 t\cos w_0(t+\tau)E[A^2] + \sin w_0 t\sin w_0(t+\tau)E[B^2]$.

$= \sigma^2\cos w_0 t\cos w_0(t+\tau) + \sigma^2\sin w_0 t + \sin w_0(t+\tau)$.

$= \sigma^2[\cos(w_0 t - w_0 t - w_0\tau)] = \sigma^2\cos(-w_0\tau) = \sigma^2\cos w_0\tau$
$= R_{xx}(\tau)$.

∴ The mean of the R.P ie $E\{x(t)\}$ = constant and the autocorrelation function is independent of absolute time 't' hence given R.P is W.S.S.

To show that $x(t)$ is not strict sense stationary we P.T $E\{x^2(t)\}$ is depends on absolute "t" consider $E[x^2(t)] = E[(A\cos w_0t + B\sin w_0t)^2]$

$= E[A^2\cos^2 w_0t + B^2\sin^2 w_0t + 2A^2B\cos^2 w_0t\sin w_0t +$
$2AB^2\cos w_0t\sin^2 w_0t)$.

$= \cos^2 w_0t\, E(A^2) + \sin^2 w_0t\, E[B^2] + 2\cos^2 w_0t\sin w_0t\, E[A^2B]$
$+ 2\cos w_0t\sin^2 w_0t\, E(AB^2)$.

Here we don't know the expected values but $E[x^2(t)]$ depends on time "t". Hence $x(t)$ is not a strict sense stationary.

→ consider the random processes $x(t) = A\cos(w_0t+\theta)$, $y(t) = B\cos(w_1t+\phi)$ where A, B, w_1 and w_0 are constants. while θ and ϕ are statistically independent random variables each uniform on $(0, 2\pi)$.

a) S.T $x(t)$ and $y(t)$ are jointly WSS.

b) If $\theta = \phi$ show that $x(t)$ and $y(t)$ are not jointly WSS unless $w_1 = w_0$.

<u>sol</u>: given that $x(t) = A\cos(w_0t+\theta)$
$y(t) = B\cos(w_1t+\phi)$.

Here θ and ϕ are statistically independent Random variables

so $f_{\theta,\phi}(\theta,\phi) = f_\theta(\theta)\cdot f_\phi(\phi)$.

and given θ, ϕ are uniformly distributed over $(0, 2\pi)$.

$f_\theta(\theta) = \begin{cases} \frac{1}{2\pi} & 0 < \theta \le 2\pi \\ 0 & \text{else} \end{cases}$, $f_\phi(\phi) = \begin{cases} \frac{1}{2\pi} & 0 < \phi \le 2\pi \\ 0 & \text{else} \end{cases}$

consider $E[x(t)] = E[A\cos(w_0t+\theta)] = A\,E[\cos(w_0t+\theta)]$.

$= A\int_{-\infty}^{\infty} \cos(w_0t+\theta) f_\theta(\theta)\, d\theta = A\int_0^{2\pi} \cos(w_0t+\theta)\cdot \frac{1}{2\pi}\, d\theta$.

$= \frac{A}{2\pi}[\sin(w_0t+\theta)]_0^{2\pi} = \underline{0}$.

$E[x(t)] = 0$.

consider $E[y(t)] = E[B\cos(w_1t+\phi)] = B\,E[\cos(w_1t+\phi)]$

$= B\int_0^{2\pi} \cos(w_1t+\phi)\cdot\frac{1}{2\pi}\, d\phi = \frac{B}{2\pi}\sin(w_1t+\phi)\Big|_0^{2\pi} = 0$

$E(y(t)) = 0$.

Consider auto correlation of Random process $x(t)$.

$R_{xx}(t, t+\tau) = E[x(t)x(t+\tau)] = E[A\cos(\omega_0 t+\theta)\, A\cos(\omega_0(t+\tau)+\theta)]$

$= A^2 E[\cos(\omega_0 t+\theta)\cos(\omega_0 t+\omega_0\tau+\theta)]$

$= \frac{A^2}{2} E[\cos(2\omega_0 t+\omega_0\tau+2\theta) + \cos\omega_0\tau]$

Now $E[\cos(2\omega_0 t+\omega_0\tau+2\theta)] = \int_{-\infty}^{\infty}\cos(2\omega_0 t+\omega_0\tau+\theta)\cdot f_\theta(\theta)\cdot d\theta$

$= \int_0^{2\pi}\cos(2\omega_0 t+\omega_0\tau+2\theta)\cdot\frac{1}{2\pi}d\theta = \frac{1}{2\pi}\left.\frac{\sin(2\omega_0 t+\omega_0\tau+2\theta)}{2}\right|_0^{2\pi}$

$= \frac{1}{4\pi}\{\sin(2\omega_0 t+\omega_0\tau) - \sin(2\omega_0 t+\omega_0\tau)\} = 0$

Similarly the auto correlation function of $y(t)$ is

$R_{yy}(t, t+\tau) = E[y(t)\, y(t+\tau)]$

$= E[B\cos(\omega_1 t+\phi)\, B\cos(\omega_1 t+\omega_1\tau+\theta)]$

$= \frac{B^2}{2} E[\cos(2\omega_1 t+2\omega_1\tau+2\phi)] + \cos(\omega_1\tau)$

$= \frac{B^2}{2}\int_0^{2\pi}\cos(2\omega_1 t+\omega_1\tau+2\phi)\frac{1}{2\pi}d\phi + \cos(\omega_1\tau)$

$= \frac{B^2}{2}\int_0^{2\pi}\underbrace{\frac{\sin(2\omega_1 t+\omega_1\tau+2\phi)}{2}}_{0}\frac{1}{2\pi}[\theta]_0^{2\pi} + \cos(\omega_1\tau)$

$= \underline{\frac{B^2}{2}\cos(\omega_1\tau)}$

Consider cross correlation function $R_{xy}(t, t+\tau) = E[x(t)\, y(t+\tau)]$

$= E[A\cos(\omega_0 t+\theta)\, B\cos(\omega_1(t+\tau)+\phi)]$

$= AB\int_0^{2\pi}\int_0^{2\pi}\cos(\omega_0 t+\theta)\cdot\cos(\omega_1 t+\omega_1\tau+\phi)\, f_\theta(\theta)\cdot f_\phi(\phi)\, d\theta\, d\phi$

$= AB\cdot\frac{1}{2\pi}\cdot\frac{1}{2\pi}\int_0^{2\pi}\int_0^{2\pi}\cos(\omega_0 t+\theta)\cos(\omega_1 t+\omega_1\tau+\phi)\, d\theta\, d\phi$

$= \frac{AB}{4\pi^2}\int_0^{2\pi}\cos(\omega_1 t+\omega_1\tau+\phi)\left.\sin(\omega_0 t+\theta)\right|_0^{2\pi} d\phi$

$= \frac{AB}{4\pi^2}\int_0^{2\pi}\cos(\omega_1 t+\omega_1\tau+\phi)\,[\sin\omega_0 t - \sin\omega_0 t] = 0$

$E[x(t)]$ and $E[y(t)]$ are constant.

The auto correlation functions $R_{xx}(t, t+\tau)$ and $R_{yy}(t, t+\tau)$ are independent of absolute time.

and also the cross correlation function $R_{xy}(t, t+\tau)$ is a function of τ [ie $R_{xy}(\tau) = 0$] Hence given o/p is jointly w.s.s.

consider $R_{xy}(t, t+\tau) = E\{x(t)\, y(t+\tau)\}$

$= E\{A\cos(\omega_0 t+\theta)\, B\cos(\omega_1 t + \omega_1 \tau + \phi)\}$

If $\theta = \phi$

$= \frac{AB}{2} E[\cos(\omega_0 t + \phi + \omega_1 t + \omega_1\tau + \phi) + \cos[\omega_0 t + \phi - \omega_1 t - \omega_1\tau - \phi]]$

$= \frac{AB}{2} E[\cos[(\omega_1+\omega_0)t + \omega_1\tau + 2\phi] + \cos[(\omega_0 - \omega_1)t - \omega_1\tau]]$

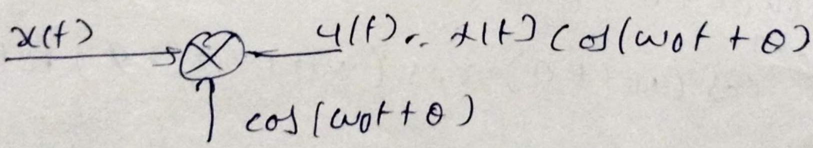
$= \frac{AB}{2}\cos[(\omega_1 - \omega_0)t - \omega_1\tau]$.

Here the cross correlation is a function of absolute time 't'. Hence given process is not jointly wss unless $\omega_1 = \omega_0$.

consider $\omega_1 = \omega_0$, The given cross correlation is a function of 'τ'.

$\therefore$ The given processes $x(t)$ and $y(t)$ are not jointly wss if $\theta = \phi$ unless $\omega_1 = \omega_0$.

$\Rightarrow$ Let $x(t)$ be a wide sense stationary random process with auto correlation function $R_{xx}(\tau) = e^{-a|\tau|}$ where $a>0$ is a constant. If $x(t)$ is amplitude modulates a carriers $\cos(\omega_0 t + \theta)$ - where ω_0 is constant and θ is random variable uniform on $(-\pi, \pi)$ ie statistically independent of $x(t)$. Determine and sketch the auto correlation function of $y(t)$.



sol: Given "θ" is a R.V distributed uniformly on $(-\pi, \pi)$ and θ is independent of $x(t)$.

auto correlation of $y(t)$ is given by

$R_{yy}(t, t+\tau) = E[y(t)\, y(t+\tau)]$

$= E[\{x(t)\cos(\omega_0 t + \theta)\}\{x(t+\tau)\cos(\omega_0(t+\tau)+\theta)\}]$

given $x(t)$ and θ are statistically independent

$R_{yy}(t, t+\tau) = E[x(t)\, x(t+\tau)]\, E[\cos(\omega_0 t + \theta) \cos(\omega_0 (t+\tau) + \theta)]$. (28)

$= R_{xx}(\tau) \cdot \frac{1}{2} E[\cos(2\omega_0 t + \omega_0 \tau + 2\theta) \cos(-\omega_0 \tau)]$.

$= \frac{1}{2} R_{xx}(\tau) \{ E[\cos(2\omega_0 t + \omega_0 \tau + 2\theta)] + E[\cos(\omega_0 \tau)] \}$.

$= \frac{1}{2} R_{xx}(\tau)\, E[\cos(2\omega_0 t + \omega_0 \tau + 2\theta)] + \frac{1}{2} R_{xx}(\tau) \cos \omega_0 \tau$ — (1)

consider $E[\cos(2\omega_0 t + \omega_0 \tau + 2\theta)] = \int_{-\infty}^{\infty} \cos(2\omega_0 t + \omega_0 \tau + 2\theta) f_\theta(\theta)\, d\theta$.

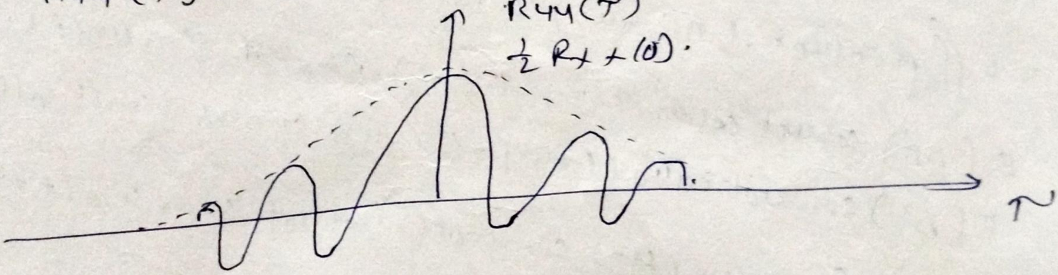
$= \int_{-\pi}^{\pi} \cos(2\omega_0 t + \omega_0 \tau + 2\theta) \frac{1}{2\pi} d\theta$.

$= \left[\frac{1}{2\pi} \cdot \frac{\sin(2\omega_0 t + \omega_0 \tau + 2\theta)}{2} \right]_{-\pi}^{\pi}$.

$= 0$

$\therefore R_{yy}(t, t+\tau) = \frac{1}{2} R_{xx}(\tau) \cos \omega_0 \tau = R_{yy}(\tau)$

$\therefore R_{yy}(\tau) = \frac{1}{2} R_{xx}(\tau) \cos \omega_0 \tau$.



→ consider two random process $x(t) = A \cos \omega_0 t + B \sin \omega_0 t$, $y(t) = B \cos \omega_0 t - A \sin \omega_0 t$. where A and B are R.V and ω_0 is a constant. Assume A and B are uncorrelated zero mean Random variables with the same variance. Find the cross correlation function $R_{xy}(t, t+\tau)$ and also show that $x(t)$ and $y(t)$ are jointly WSS.

sol: consider $E[x(t)] = E[A] \cdot \cos \omega_0 t + E[B] \sin \omega_0 t = 0$

consider $R_{xx}(t, t+\tau) = E[x(t) \cdot x(t+\tau)]$.

$= E[(A \cos \omega_0 t + B \sin \omega_0 t)(A \cos \omega_0 (t+\tau) + B \sin(\omega_0 (t+\tau))]$.

$= E[A^2 \cos \omega_0 t \cos \omega_0 (t+\tau) + AB \cos \omega_0 t \sin \omega_0 (t+\tau) + AB \sin \omega_0 t \cos \omega_0 (t+\tau) + B^2 \sin \omega_0 t \sin \omega_0 (t+\tau)]$.

$= E(A^2)\cos\omega_0 t \cos\omega_0(t+\tau) + E(AB)\sin(2\omega_0 t + \omega_0\tau) +$
$E(B^2)\sin\omega_0 t \sin\omega_0(t+\tau)$

since A and B are zero mean, $E(A^2) = var(A)$ and
$E(B^2) = var(B)$

since A and B are uncorrelated $\rho_{AB} = \frac{cov(A,B)}{\sigma_A \sigma_B} = 0 \Rightarrow$

$cov(A,B) = 0$

$\Rightarrow E(AB) = E(A)E(B) = 0$

$\therefore R_{xx}(t,t+\tau) = E(A^2)\cos(\omega_0 t\cos(\omega_0(t+\tau)) + \sin\omega_0 t \sin(\omega_0(t+\tau))$

$= E(A^2)\cos\omega\tau$

$\Rightarrow x(t)$ is a stationary process, similarly $y(t)$ is also stationary. can be verified like stationary.

consider $R_{xy}(t,t+\tau) = E\{x(t)y(t+\tau)\}$

$= E\{(A\cos\omega_0 t + B\sin\omega_0 t)(B\cos(\omega_0(t+\tau)) - A\sin\omega_0(t+\tau))\}$

$= E(AB)\cos\omega_0 t\cos\omega_0(t+\tau) - E(A^2)\cos\omega_0 t \sin\omega_0(t+\tau) +$
$E(B^2)\sin\omega_0 t\cos\omega_0(t+\tau) - E(AB)\sin\omega_0 t\sin(\omega_0(t+\tau))$

$= E(A^2)\{\cos(\omega_0(t+\tau))\sin\omega_0 t - \cos\omega_0 t\sin\omega_0(t+\tau)\}$

$= E(A^2)\sin\omega\tau$.

since $R_{xy}(t,t+\tau)$ is independent of time and is a function of τ, $x(t)$ and $y(t)$ are jointly stationary processes

<u>correlation functions:-</u>

Basically there are two correlation functions

(1) Auto correlation function

(2) cross correlation function

<u>NOTE</u>:- Auto correlation measures the similarity of same process $x(t)$
i.e $R_{xx}(t,t+\tau) = E[x(t)x(t+\tau)]$.

cross correlation is used to measure the similarity of two different process $x(t)$ and $y(t)$ i.e $R_{xy}(t,t+\tau) = E\{x(t)y(t+\tau)\}$.

Auto correlation function :-

Let us consider a random process $X(t)$. The auto correlation function of a $X(t)$ is the correlation of two random variable $X_1 = X(t_1)$ and $X_2 = X(t_2)$ defined by the process of t_1 & t_2.

Mathematically

$$R_{XX}(t_1, t_2) = E[X(t_1) \cdot X(t_2)] \quad - (1)$$

Let $t_1 = t$ and $t_2 = t + T$ with T is a real number

$$R_{XX}(t, t+T) = E\{X(t)\, X(t+T)\} - (2)$$

If $X(t)$ is atleast w.s.s process then $R_{XX}(t, t+T)$ is a function of T only hence $R_{XX}(T) = E\{X(t)\, X(t+T)\} - (3)$

Properties :- (1) The mean square value of a random process is obtained by putting $T = 0$ ie $R_{XX}(0) = E\{X^2(t)\} \rightarrow$ avg Power

Proof. we know that $R_{XX}(T) = E[X(t)\, X(t+T)]$

put $T = 0$ then $R_{XX}(0) = E[X(t)\, X(t+0)] = E[X^2(t)]$

Property 2 :-

$R_{XX}(T)$ is an even function of T ie $R_{XX}(-T) = R_{XX}(T)$

Proof :- w.k.T $R_{XX}(T) = E[X(t)\, X(t+T)]$

$R_{XX}(-T) = E[X(t)\, X(t-T)]$

now let $t - T = u \Rightarrow t = u + T$

$\Rightarrow R_{XX}(-T) = E\{X(u+T)\, X(u)\} = E[X(u)\, X(u+T)]$

$= R_{XX}(T)$.

Property 3 :-

The maximum value of auto correlation function occurs at the origin ie $|R_{XX}(T)| \leq R_{XX}(0)$.

Proof :- Let us consider $E[(X(t_1) \pm X(t_2))^2] \geq 0$

$\Rightarrow E[X^2(t_1) + X^2(t_2) \pm 2X(t_1)\, X(t_2)] \geq 0$

$\Rightarrow E[X^2(t_1)] + E[X^2(t_2)] \pm 2E[X(t_1)\, X(t_2)] \geq 0$

$\Rightarrow R_{XX}(0) + R_{XX}(0) \pm 2E[X(t_1)\, X(t_2)] \geq 0$

$\Rightarrow R_{XX}(0) + R_{XX}(0) \pm 2 \cdot R_{XX}(t_1, t_2) \geq 0$

$\Rightarrow 2 \cdot R_{xx}(0) \pm 2 R_{xx}(t_1 t_2) \geq 0$

If $x(t)$ is a w.s.s then auto correlation is a function of "τ"

$\Rightarrow 2 R_{xx}(0) \pm 2 R_{xx}(\tau) \geq 0$

$\Rightarrow 2 R_{xx}(0) \geq 2 |R_{xx}(\tau)|$

$\Rightarrow R_{xx}(0) \geq |R_{xx}(\tau)|$.

$\therefore$ The maximum value of auto correlation occurs at the origin ie $R_{xx}(\tau)$ is bounded by it's value at the origin

<u>Property 4:-</u>

If a random process $x(t)$ has non-zero mean value ie $E[x(t)] = \bar{x} \neq 0$ and ergodic has no periodic components then

$$\lim_{|\tau| \to \infty} R_{xx}(\tau) = \bar{x}^2$$

<u>proof:-</u> consider a random process $x(t)$ with R.V.s $x(t_1)$ and $x(t_2)$

now the auto correlation $R_{xx}(\tau) = E[x(t_1)\, x(t_2)]$.

since the process has no periodic components as $|\tau| \to \infty$. The random variables becomes independent

Then $\lim_{|\tau| \to \infty} R_{xx}(\tau) = E\{x(t_1)\} E\{x(t_2)\}$

given $x(t)$ is ergodicity then

$E\{x(t_1)\} = E\{x(t_2)\} = \bar{x}$

Then $\therefore \lim_{|\tau| \to \infty} R_{xx}(\tau) = \bar{x} \cdot \bar{x} = \bar{x}^2$

<u>Property 5:-</u> If $x(t)$ has a periodic component then $R_{xx}(\tau)$ can also periodic components with the same period

<u>proof:-</u> consider auto correlation function

$R_{xx}(\tau) = E[x(t)\, x(t+\tau)]$

$R_{xx}(\tau \pm T_0) = E[x(t)\, x(t+\tau \pm T_0)]$

given $x(t)$ is a periodic then

$x(t+\tau \pm T_0) = x(t+\tau)$

$\Rightarrow R_{xx}(\tau \pm T_0) = E[x(t)\, x(t+\tau)] = R_{xx}(\tau)$.

$R_{xx}(\tau)$ is periodic with the same period "T_0".

Property 6 :-

If $x(t)$ is ergodic and zero mean value and has no periodic component then $\lim_{|\tau| \to \infty} R_{xx}(\tau) = 0$

Proof :-

w.k.T $\lim_{|\tau| \to \infty} R_{xx}(\tau) = \bar{x}^2$

Given process $x(t)$ has zero mean value i.e $\bar{x} = 0$

$\Rightarrow \lim_{|\tau| \to \infty} R_{xx}(\tau) = 0$

Property 7 :- Auto correlation function $R_{xx}(\tau)$ can not have an arbitrary shape.

NOTE :- The avg power of the r.P equals to the $R_{xx}(0)$. i.e

Avg. Power $= R_{xx}(0) = E\{x^2(t)\}$ watts.

→ **Problems :-** A stationary ergodic random process has the autocorrelation function with no periodic components is $R_{xx}(\tau) = 25 + \frac{4}{1+6\tau^2}$

Find the mean and variance of the process $x(t)$.

Sol :- Given $x(t)$ is ergodic and has no periodic components.

$\Rightarrow Lt_{|\tau| \to \infty} R_{xx}(\tau) = \bar{x}^2$

$\bar{x}^2 = Lt_{|\tau| \to \infty} \left(25 + \frac{4}{1+6\tau^2}\right) = 25 + Lt_{|\tau| \to \infty} \frac{4}{1+6\tau^2}$

$= 25 + \frac{4}{\infty} = \underline{25}.$

$\bar{x} = \underline{\pm 5}$

→ variance of the $x(t)$ is $= E\left[x^2(t)\right] - \left(\overline{x(t)}\right)^2$

consider $\left(x^2(t)\right) = R_{xx}(0) = 25 + \frac{4}{1+0} = 29.$

variance $= 29 - (5)^2 = \underline{4}.$

→ A stationary process has an autocorrelation function is given by

$R_{xx}(\tau) = \frac{25\tau^2 + 36}{6.25\tau^2 + 4}$ Find the mean square value of the process

Sol :- mean value $Lt_{|\tau| \to \infty} R_{xx}(\tau) = \bar{x}^2$

$\Rightarrow \bar{x}^2 = \lim_{|\tau| \to \infty} \frac{25\tau^2 + 36}{6.25\tau^2 + 4}$

$$= \lim_{|\tau| \to \infty} \frac{25 + \frac{36}{\tau^2}}{6.25 + \frac{4}{\tau^2}} = 4$$

$\bar{X} = \pm 2$

→ mean square value $E[X^2(t)] = R_{XX}(0)$

$E[X^2(t)] = \frac{36}{4} = 9.$

→ variance $= E[X^2(t)] - \bar{X}^2 = 9 - 4 = 5.$

⇒ Assume that an ergodic Random process $X(t)$ has an auto correlation function $R_{XX}(\tau) = 18 + \frac{2}{6+\tau^2}(1+4\cos(2\tau))$.
Find (i) $|\bar{X}|$ (ii) Find the ~~[illegible]~~ avg power on $X(t)$.
(iii) Does this process have an periodic components.

Sol:- $R_{XX}(\tau) = 18 + \frac{2}{6+\tau^2}(1+4\cos(2\tau))$

$$\bar{X}^2 = \lim_{\tau \to \infty} R_{XX}(\tau)$$

(i) Mean value $|\bar{X}^2| = \lim_{\tau \to \infty} |R_{XX}(\tau)| \Rightarrow |R_{XX}(\tau)| = 18 + \frac{2}{6+\tau^2}(1+4)$

$= 18 + \frac{10}{6+\tau^2}$

$|\bar{X}|^2 = \lim_{\tau \to \infty} 18 + \frac{10}{6+\tau^2} \Rightarrow |\bar{X}|^2 = 18$

$|\bar{X}| = \sqrt{18} = 4.2426$

(ii) For a periodic component

$$\lim_{\tau \to \infty} R_{XX}(\tau) = \lim_{\tau \to \infty} \left(18 + \frac{2}{6+\tau^2}\right)(1+4\cos(2\infty))$$

$= \infty$

This process has ~~[illegible]~~ periodic compnent

(iii) Avg Power of $X(t)$ is $E[X^2] = R_{XX}(0) = 18 + \frac{2}{6}(1+4\cos 0)$

$= 18 + \frac{10}{6} = 19.667.$

Cross correlation function

The cross correlation function of a two random process $x(t)$ and $y(t)$ is defined as $R_{xy}(t, t+\tau) = E\{x(t)\, y(t+\tau)\}$ — (1).

If $x(t)$ and $y(t)$ are atleast w.s.s the cross relation is a function of $\tau \Rightarrow R_{xy}(\tau) = E\{x(t)\, y(t+\tau)\}$

Properties:

1. cross correlation satisfies the symmetry property ie $R_{yx}(\tau) =$

$$R_{xy}(\tau) = R_{yx}(-\tau)$$

we know that the cross correlation of two process $x(t)$ & $y(t)$ is given by $R_{xy}(\tau) = E[x(t)\, y(t+\tau)]$

$$\Rightarrow R_{yx}(\tau) = E[y(t)\, x(t+\tau)]$$

$$\Rightarrow R_{yx}(-\tau) = E\{y(t)\, x(t-\tau)\}$$

put $t-\tau = u$ and $t = u+\tau$

$$R_{yx}(-\tau) = E[y(u+\tau)\, x(u)]$$

$$= E[x(u)\, y(u+\tau)] = R_{xy}(\tau)$$

$\Rightarrow$ If $R_{xx}(\tau)$ and $R_{yy}(\tau)$ are the auto correlation functions of $x(t)$ and $y(t)$ respectively. then the cross correlation is given by $|R_{xy}(\tau)| \le \sqrt{R_{xx}(0)\, R_{yy}(0)}$

Proof: consider $E\left[\left(\frac{x(t_1)}{\sqrt{R_{xx}(0)}} \pm \frac{y(t_2)}{\sqrt{R_{yy}(0)}}\right)^2\right] \ge 0$

$$\Rightarrow E\left[\frac{x^2(t_1)}{R_{xx}(0)} + \frac{y^2(t_2)}{R_{yy}(0)} \pm 2\frac{x(t_1)y(t_2)}{\sqrt{R_{xx}(0)}\cdot\sqrt{R_{yy}(0)}}\right] \ge 0$$

$$= \frac{E\{x^2(t_1)\}}{R_{xx}(0)} + \frac{E\{y^2(t_2)\}}{R_{yy}(0)} \pm \frac{2E(x(t_1)\cdot y(t_2))}{\sqrt{R_{xx}(0)\cdot R_{yy}(0)}} \ge 0$$

$$= \frac{R_{xx}(0)}{R_{xx}(0)} + \frac{R_{yy}(0)}{R_{yy}(0)} \pm \frac{2E(x(t_1)y(t_2))}{\sqrt{R_{xx}(0)\, R_{yy}(0)}} \ge 0$$

$$\Rightarrow \frac{2 + 2E\left[\left(x(t_1)\, y(t_2)\right)\right]}{\sqrt{R_{xx}(0) \cdot R_{yy}(0)}} \geq 0$$

If two process are jointly w.s.s then cross corelation is a function of τ $2 \geq 2 \frac{|R_{xy}(\tau)|}{\sqrt{R_{xx}(0) R_{yy}(0)}}$

$$\Rightarrow \sqrt{R_{xx}(0) R_{yy}(0)} \geq |R_{xy}(\tau)|$$

$$\Rightarrow |R_{xy}(\tau)| \leq \sqrt{R_{xx}(0) \cdot R_{yy}(0)}$$

Property ③ :: The geometric mean two positive Quantities cannot exceeds their arithmatic mean.

Solution: If $x(t)$ and $y(t)$ are random processes then

$$|R_{xy}(\tau)| \leq \frac{1}{2}\left[R_{xx}(0) + R_{yy}(0)\right]$$

we know that $|R_{xy}(\tau)| \leq \sqrt{R_{xx}(0) R_{yy}(0)}$ and — ①

$$\sqrt{R_{xx}(0) \cdot R_{yy}(0)} \leq \frac{R_{xx}(0) + R_{yy}(0)}{2} \quad — ②$$

From ① & ②

$$|R_{xy}(\tau)| \leq \frac{R_{xx}(0) + R_{yy}(0)}{2}$$

4. If random process $x(t)$ and $y(t)$ are independent and atleast w.s.s then $R_{xy}(\tau) = \bar{x} \cdot \bar{y}$

proof: If $x(t)$ and $y(t)$ are statistically independent and jointly w.s.s then $R_{xy}(\tau) = E[x(t)] \cdot E[y(t+\tau)]$

$$R_{xy}(\tau) = \bar{x} \cdot \bar{y}$$

⇒ If the random processes $x(t)$ and $y(t)$ have zero mean and wide sense stationary then $\lim_{|\tau| \to \infty} R_{xy}(\tau) = 0$

Proof: $\lim_{|\tau| \to \infty} R_{xy}(\tau) = \lim_{|\tau| \to \infty} E[x(t)\, y(t+\tau)]$

As $|T| \to \infty$ two process $x(t)$ and $y(t)$ are independent

$\Rightarrow \lim_{|T| \to \infty} R_{xy}(T) = E[x(t)]\, E[y(t+T)]$

$= \bar{x} \cdot \bar{y} = 0 \cdot 0 = 0$

Auto covariance

Auto covariance of a random process $x(t)$ is

$$C_{xx}(t, t+T) = R_{xx}(t, t+T) - E[x(t)]\, E[x(t+T)] \quad -(1)$$

If $x(t)$ is atleast w.s.s then

$$C_{xx}(T) = R_{xx}(T) - \bar{x}^2$$

Cross-covariance :-

The cross covariance of two random process $x(t)$ and $y(t)$ is given by

$$C_{xy}(t, t+T) = R_{xy}(t, t+T) - E[x(t)]\, E[y(t+T)] \quad -(2)$$

If $x(t)$ and $y(t)$ are atleast joint w.s.s then

$$C_{xy}(T) = R_{xy}(T) - \bar{x}\bar{y} \quad -(3)$$

→ If the two random processes are independent then $C_{xy}(T) = 0$.

⇒ Consider a R.P $x(t) = A\cos\omega t$ where ω is constant and A is R.V with uniformly distributed over an (0,1). Find (i) auto correlation (ii) Find auto covariance

sol: Given $x(t) = A\cos\omega t$ where ω is constant & A is R.V

$$f_A(A) = \begin{cases} 0 & 0 \le A \le 1 \\ 0 & \text{else} \end{cases}$$

→ Auto correlation $R_{xx}(t, t+T) = E[x(t)\, x(t+T)]$

$= E[A\cos\omega t \cdot A\cos(\omega(t+T))] = \cos\omega t \cos\omega(t+T)\, E[A^2]$

$= \cos\omega t \cdot \cos\omega(t+T) \int_0^1 A^2 \cdot 1 \, dA$

$= \cos\omega t \cos\omega(t+T) \cdot \frac{A^3}{3}\Big|_0^1 = \frac{1}{3} \cos\omega t \cos\omega(t+T)$

wrong

Auto co-variance

$$C_{xx}(t, t+T) = R_{xx}(t, t+T) - E\{x(t)\}E[x(t+T)]$$

$$= \frac{1}{3}\cos\omega t \cos\omega(t+T) - E\{x(t)\}E(x(t+T)\}.$$

$$E[x(t)] = E[A\cos\omega t] = \cos\omega t\, E[A] = \cos\omega t \int_0^1 A \cdot dA =$$

$$= \frac{1}{2}\cos\omega t.$$

$$E[x(t+T)] = E[A\cos(\omega(t+T))] = \cos\omega(t+T)\, E(A)$$

$$= \cos\omega(t+T) \cdot \frac{1}{2}.$$

$$C_{xx}(t, t+T) = \frac{1}{3}\cos\omega t \cos\omega(t+T) - \frac{1}{4}\cos\omega t \cos\omega(t+T)$$

$$= \frac{\cos\omega t \cos\omega(t+T)}{12}$$

Covariance functions for random Processes

Auto covarience function

The " " " is a measure of inter dependence b/w two random variables.

consider random processes $x(t)$, $x(t+T)$ at two time intervals t and $t+T$. The auto covarience function defined and can be expressed as

$$C_{xx}(t, t+T) = E\left[\left(x(t) - E(x(t))\right)\left(x(t+T) - E(x(t+T))\right)\right]$$

$$= E\left[x(t) \cdot x(t+T)\right] - E[~~x(t) E[x(t+T)~~]]$$

$$- E[x(t)] E[x(t+T)] + E[~~x(t) E[x(t+T)~~]]$$

$$C_{xx}(t, t+T) = E[x(t) \cdot x(t+T)] - E[x(t)] E[x(t+T)]$$

$$C_{xx}(t, t+T) = R_{xx}(t, t+T) - E[x(t)] E[x(t+T)]$$

NOTE 1:: If $x(t)$ is atleast a W.S.S.R.P, then mean is constant, auto correlation function only on time difference.

$$C_{xx}(T) = R_{xx}(T) - \bar{x}^2$$

The auto covarience function is not function of absolute time 't'.

NOTE 2:: At $T=0$

$$C_{xx}(0) = R_{xx}(0) - \bar{x}^2$$

$$= E[x^2] - \bar{x}^2 \quad [\because R_{xx}(0) = E[x^2]]$$

$C_{xx}(0) = \sigma_x^2$

∴ At $\tau = 0$, The auto covarience function becomes the varience of the random process.

Note 3:

The auto correlation coefficient of random process $x(t)$ is defined as

$$\rho_{xx}(t, t+\tau) = \frac{C_{xx}(t, t+\tau)}{\sqrt{C_{xx}(t,t) \cdot C_{xx}(t+\tau, t+\tau)}}$$

At for WSS process

$$\rho_{xx}(\tau) = \frac{C_{xx}(\tau)}{\sqrt{C_{xx}(\tau) \cdot C_{xx}(\tau)}}$$

At $\tau = 0$

$$\rho_{xx}(0) = \frac{C_{xx}(0)}{\sqrt{C_{xx}(0) \cdot C_{xx}(0)}}$$

$$\boxed{\rho_{xx}(0) = 1}$$

Cross covarience function

If two random processes $x(t)$, $y(t)$ have random variables at $x(t)$ and $y(t+\tau)$ then the cross covarience function is defined as

$$C_{xy}(t, t+\tau) = E\left[\left(x(t) - E[x(t)]\right) \cdot \left(y(t+\tau) - E[y(t+\tau)]\right)\right]$$

$$C_{xy}(t, t+T) = E[x(t) \cdot y(t+T)] - E[x(t)]E[y(t+T)] - E[x(t)E(y(t+T))] + E[~~x(t)~~ ~~E~~ \{y(t+T)\}]$$

$$C_{xy}(t, t+T) = R_{xy}(t, t+T) - E[x(t)]\, E[y(t+T)] \quad -①$$

NOTE1: If the random processes are atleast jointly WSS then $C_{xy}(t, t+T)$ is

$$C_{xy}(T) = R_{xy}(T) - \bar{x} \cdot \bar{y}$$

The cross covarience function is not a function of absolute time t!"

NOTE2: If two random processes $x(t)$ and $y(t)$ ~~not~~ are uncorrelated then $C_{xy}(t, t+T) = 0 \quad -②$

substitute eq ② in eq ① Then

$$R_{xy}(t, t+T) = E[x(t)]\, E[y(t+T)] \quad -⑦$$

This is the condition for two random processes to be statistically independent, therefore the independent random processes are uncorrelated.

Note 3: The cross correlation coefficient of random processes $x(t)$ and $y(t)$ is defined as

$$\rho_{xy}(t, t+T) = \frac{C_{xy}(t, t+T)}{\sqrt{C_{xx}(t, t) \cdot C_{yy}(t+T, t+T)}}$$

At $T = 0$

$$\rho_{xx}(0) = \frac{C_{xy}(0)}{\sqrt{C_{xx}(0) \cdot C_{yy}(0)}}$$

→ Consider a random process $x(t) = A\cos\omega t$ where ω is a constant and A is random variable uniformly distributed over (0,1). Find auto correlation, auto covariance of $x(t)$.

Solu: Given $x(t) = A\cos\omega t$

i) $R_{xx}(T) = E[x(t) \cdot x(t+T)] = E[A\cos\omega t \cdot A\cos\omega(t+T)]$

$= E[A^2 \cos\omega t \cdot \cos(\omega t + \omega T)]$

$= \int_0^1 A^2 \cos\omega t \cdot \cos(\omega t + \omega T) \cdot f_A(A) \cdot dA$

since "A" is uniformly distributed random variable

$\therefore f_A(A) = \begin{cases} 1 & 0 \le A \le 1 \\ 0 & \text{else} \end{cases}$

$R_{xx}(T) = \int_0^1 A^2 \cos\omega t \cdot \cos(\omega t + \omega T) \cdot 1 \cdot dA$

$= \cos\omega t \cdot \cos(\omega t + \omega T) \cdot \frac{A^3}{3}\Big|_0^1$

$R_{xx}(T) = \frac{\cos\omega t \cdot \cos(\omega t + \omega T)}{3}$

ii) Auto covariance

$C_{xx}(t, t+T) = R_{xx}(T) - E(x(t)) \cdot E[x(t+T)]$

$E[x(t)] = \int_0^1 x(t) \cdot f_A(A) \cdot dA$

$= \int_0^1 A \cdot \cos\omega t \cdot 1 \cdot dA = \cos\omega t \cdot \frac{A^2}{2}\Big|_0^1$

$E(x(t)) = \frac{\cos\omega t}{2}$ similarly $E[x(t+T)] = \frac{\cos(\omega(t+T))}{2}$

$C_{xx}(t, t+T) = \frac{\cos\omega t \cdot \cos(\omega t + \omega T)}{3} - \frac{\cos\omega t \cdot \cos(\omega t + \omega T)}{4}$

$= \frac{\cos\omega t \cdot \cos\omega(t+T)}{12}$

Gaussian Random processes

Consider a continuous random process $X(t)$;

Let N random variables $X_1 = X(t_1), X_2 = X(t_2) \ldots X_N = X(t_N)$ defined at time instants $t_1, t_2, t_3, \ldots t_N$ respectively.

If these random variables are jointly Gaussian for any $N = 1, 2 \ldots$ and at anytime instants $t_1, t_2 \ldots t_N$, then the random process process $X(t)$ is called Gaussian Random processes

The joint density function is given as

$$f_X(x_1, x_2 \ldots x_N; t_1, t_2 \ldots t_N) = \frac{\exp\left\{-\frac{1}{2}[x - \bar{X}]^t [C_X]^{-1} [X - \bar{X}]\right\}}{\sqrt{(2\pi)^N |[C_X]|}}$$

where $\bar{X} = E[x_i] = E[X(t)]$

$[C_X]$ = covariance matrix and its elements are

$$C_{ik} = C_{x_i x_k} = E[(x_i - \bar{X_i})(x_k - \bar{X_k})]$$

$$= E[(x(t_i) - \overline{X(t_i)})(x(t_k) - \overline{X(t_k)})]$$

$$C_{ik} = C_{XX}(t_i, t_k)$$

C_{ik} is the autocovariance of $(X t_i)$ and $(X t_k)$

Also by expanding the above equation we can get

$$C_{XX}(t_i, t_k) = R_{XX}(t_i, t_k) - E[X(t_i)]E[X(t_k)]$$

where $R_{XX}(t_i, t_k)$ is the autocorrelation function of X

if the process is wide-sense stationary then

i) Mean value will be constant i.e;

$$\bar{X_i} = E[X(t_i)] = \bar{X} \text{ constant}$$

2) The mean autocorrelation and autocovariance function will depend only on time differences and not on absolute time

$$C_{XX}(t_i, t_k) = C_{XX}(t_k - t_i)$$

$$R_{XX}(t_i, t_k) = R_{XX}(t_k - t_i)$$

Poisson Random processes

The poisson process x(t) is a discrete random process, which represents the number of times that some event has occurred as a function of time, x(t) has integer-valued nondecreasing sample functions. Such as check in register, arrival of a customer, arrival of vehicles at a particular point etc

In these functions a single event occurs at a random time, counting the number of occurrences with time is a poisson process. It is also called counting process. shows the sample function of a poisson counting processes

The conditions for Poisson process x(t) are

1) $x(0) = 0$

2) The event occurs only one in any instant of time i.e infinitesimal time interval.

3) The number that occurs in any given time intervals is independent of the number in any other non overlapping time interval

i.e, x(t) has independent increments

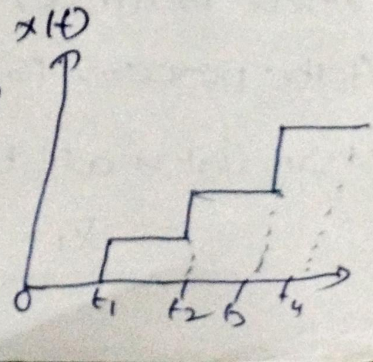


Fig. Poission counting processes

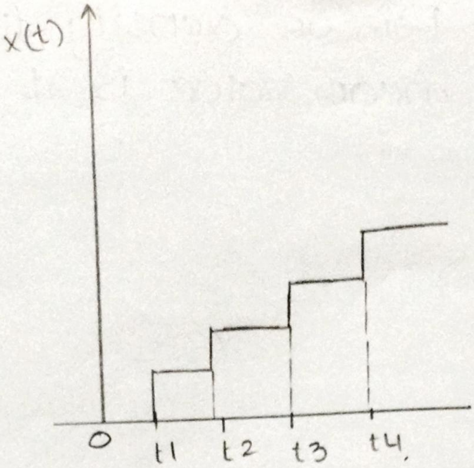


Fig 6.3: poisson counting processes

Probability density function

If the number of occurances of an event ~~occurrences~~ in any finite interval of time, that is described by Poisson distribution and the average rate of occurrencess is λ then, the probability of exactly k occurrences over a time interval (0, t) is

$$p[x(t) = k] = \frac{(\lambda t)^k e^{-\lambda t}}{k!}, \quad k = 0, 1, 2 \ldots$$

and the probability density function

$$f_X(x) = \sum_{k=0}^{\infty} \frac{(\lambda t)^k e^{-\lambda t}}{k!} \delta(x - k)$$

Mean value:-

The mean value of poisson density function is

$$E[x(t)] = \int_{-\infty}^{\infty} x f_X(x) dx = \int_{-\infty}^{\infty} x \sum_{k=0}^{\infty} \frac{(\lambda t)^k e^{-\lambda t}}{k!} \delta(x-k) dx$$

since $\int_{-\infty}^{\infty} x \delta(x-k) dx = k$

$$\therefore E[x(t)] = \sum_{k=0}^{\infty} \frac{k (\lambda t)^k e^{-\lambda t}}{k!}$$

But we know that, from Poisson density function of a random variable the mean value is λt

So, $E[X(t)] = \lambda t$

$$\therefore \sum_{k=0}^{\infty} \frac{k(\lambda t)^k e^{-\lambda t}}{k!} = \lambda t$$

Second moment:-

The second moment is

$$E[X^2(t)] = \int_{-\infty}^{\infty} x^2 f_X(x) dx$$

$$\int_{-\infty}^{\infty} x^2 \sum_{k=0}^{\infty} \frac{(\lambda t)^k e^{-\lambda t}}{k!} \delta(x-k) dx$$

since $\int_{-\infty}^{\infty} x^2 \delta(x-k) dx = k^2$

$$E[X^2(t)] = \sum_{k=0}^{\infty} \frac{k^2(\lambda t)^k e^{-\lambda t}}{k!}$$

But we know that, the second moment is

$$E[X^2] = \sigma_x^2 + \bar{X}^2$$

$$= \lambda t + (\lambda t)^2$$

$$= \lambda t + \lambda^2 t^2$$

$$E[X^2(t)] = \sum_{k=0}^{\infty} \frac{k^2(\lambda t)^k e^{-\lambda t}}{k!} = \lambda t + \lambda^2 t^2$$