UNIT-I
FRICTION

FRICTION:

when ever one block moves or tends to move tangentially with respect to the surface, on which it rests, the interlocking property of the projecting particles opposes the motion. This opposing force, which acts in the opposite direction of the movement of the upper block, is called the force of friction or simply friction.

Types of Friction: In general, the friction is of the following two types :

1. *Static friction.* It is the friction, experienced by a body, when at rest.

2. *Dynamic friction.* It is the friction, experienced by a body, when in motion. The dynamic friction is also called ***kinetic friction*** and is less than the static friction.
It is of the following three types :

(*a*) ***Sliding friction.*** It is the friction, experienced by a body , when it ***slides*** over another body.

(*b*) ***Rolling friction.*** It is the friction, experienced between the surfaces which has ***balls*** or ***rollers*** interposed between them.

(*c*) ***Pivot friction.*** It is the friction, experienced by a body, due to the ***motion of rotation*** as in case of foot step bearings.

The friction may further be classified as :

10.3. Friction Between Unlubricated Surfaces

The friction experienced between two dry and unlubricated surfaces in contact is known as ***dry*** or ***solid friction***.

10.4. Friction Between Lubricated Surfaces

When lubricant (*i.e.* oil or grease) is applied between two surfaces in contact, then the friction may be classified into the following two types depending upon the thickness of layer of a lubricant.

1. *Boundary friction* (*or greasy friction or non-viscous friction*). It is the friction, experienced between the rubbing surfaces, when the surfaces have a very thin layer of lubricant.

2. *Fluid friction* (*or film friction or viscous friction*). It is the friction, experienced between the rubbing surfaces, when the surfaces have a thick layer of the lubrhicant.

10.5. Limiting Friction

The maximum value of frictional force, which comes into play, when a body just begins to slide over the surface of the other body, is known as limiting force of friction or simply limiting friction.

10.6. Laws of Static Friction

Following are the laws of static friction :

1. The force of friction always acts in a direction, opposite to that in which the body tends to move.
2. The magnitude of the force of friction is exactly equal to the force, which tends the body to move.
3. The magnitude of the limiting friction (F) bears a constant ratio to the normal reaction (R_N) between the two surfaces. Mathematically

$$F/R_N = \text{constant}$$

4. The force of friction is independent of the area of contact, between the two surfaces.

5.The force of friction depends upon the roughness of the surfaces.

COEFFICIENT OF FRICTION: It is defined as the ratio of the limiting friction (F) to the normal reaction (RN) between the two bodies.

$$\mu = F/RN$$

Example 10.1. *A body, resting on a rough horizontal plane required a pull of 180 N inclined at 30º to the plane just to move it. It was found that a push of 220 N inclined at 30º to the plane just moved the body. Determine the weight of the body and the coefficient of friction.*

Solution. Given : $\theta = 30º$

Let W = Weight of the body in newtons,

R_N = Normal reaction,

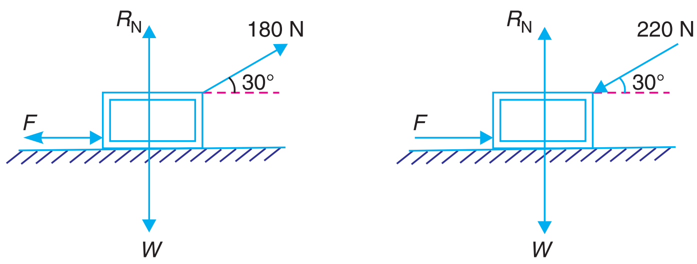
μ = Coefficient of friction, and

F = Force of friction.

First of all, let us consider a pull of 180 N. The force of friction (F) acts towards left as shown in Fig. 10.5 (a).

Resolving the forces horizontally,

$$F = 180 \cos 30º = 180 \times 0.866 = 156 \text{ N}$$



(a) (b)

Fig. 10.5

Now resolving the forces vertically,

$$R_N = W - 180 \sin 30º = W - 180 \times 0.5 = (W - 90) \text{ N}$$

We know that $F = \mu . R_N$ or $156 = \mu\,(W - 90)$ ...(*i*)

Now let us consider a push of 220 N. The force of friction (F) acts towards right as shown in Fig. 10.5 (b).

Resolving the forces horizontally,

$$F = 220 \cos 30º = 220 \times 0.866 = 190.5 \text{ N}$$

Now resolving the forces vertically,

$$R_N = W + 220 \sin 30º = W + 220 \times 0.5 = (W + 110) \text{ N}$$

We know that $F = \mu . R_N$ or $190.5 = \mu\,(W + 110)$ ...(*ii*)

From equations (*i*) and (*ii*),

W = 1000 N, and $\mu = 0.1714$ **Ans.**

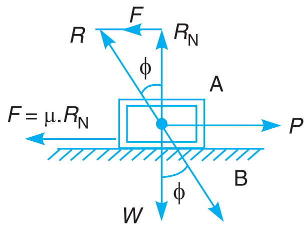
10.14. Friction of a Body Lying on a Rough Inclined Plane

∴

ANGLE OF FRICTION:

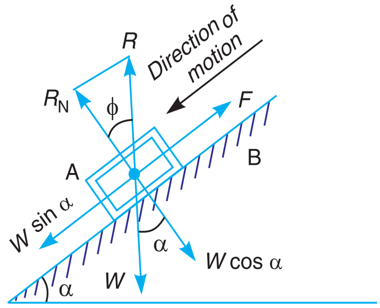
It may be defined as the angle which the resultant reaction R makes with the normal reaction RN.

tan φ = F/RN = μRN/RN = μ



ANGLE OF REPOSE:

If the angle of inclination α of the plane to the horizontal is such that the body begins to move down the plane, then the angle α is called the angle of repose. A little consideration will show that the body will begin to move down the plane when the angle of inclination of the plane is equal to the angle of friction (i.e. α = φ).



10.14. Friction of a Body Lying on a Rough Inclined Plane

1. *Considering the motion of the body up the plane*

Let W = Weight of the body,

α = Angle of inclination of the plane to the horizontal,

ϕ = Limiting angle of friction for the contact surfaces,

P = Effort applied in a given direction in order to cause the body to slide with uniform velocity parallel to the plane, considering friction,

P_0 = Effort required to move the body up the plane neglecting friction,

θ = Angle which the line of action of P makes with the weight of the body W,

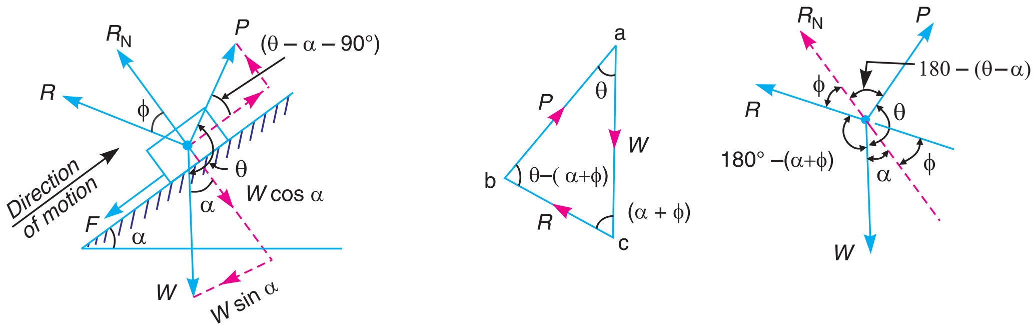
μ = Coefficient of friction between the surfaces of the plane and the body,

R_N = Normal reaction, and

R = Resultant reaction.

When friction is taken into account, a frictional force $F = \mu.R_N$ acts in the direction opposite to the motion of the body, as shown in Fig. 10.8 (*a*). The resultant reaction R between the plane and the body is inclined at an angle ϕ with the normal reaction R_N. The triangle of forces is shown in Fig. 10.8 (*b*). Now applying sine rule,

$$\frac{P}{\sin(\alpha + \phi)} = \frac{W}{\sin[\theta - (\alpha + \phi)]}$$



(*a*) (*b*) (*c*)

Fig. 10.8. Motion of the body up the plane, considering friction.

$$\therefore \qquad P = \frac{W \sin(\alpha + \phi)}{\sin[\theta - (\alpha + \phi)]} \qquad \ldots(ii)$$

Notes : 1. When the effort applied is horizontal, then $\theta = 90°$. In that case, the equations (*i*) and (*ii*) may be written as

$$P = \frac{W\sin(\alpha+\phi)}{\sin[90^\circ-(\alpha+\phi)]} = \frac{W\sin(\alpha+\phi)}{\cos(\alpha+\phi)} = W\tan(\alpha+\phi)$$

2. When the effort applied is parallel to the plane, then $\theta = 90^\circ + \alpha$. In that case, the equations ***(i)*** and ***(ii)*** may be written as

$$P = \frac{W\sin(\alpha+\phi)}{\sin[(90^\circ+\alpha)-(\alpha+\phi)]} = \frac{W\sin(\alpha+\phi)}{\cos\phi}$$

$$= \frac{W(\sin\alpha\cos\phi+\cos\alpha\sin\phi)}{\cos\phi} = W(\sin\alpha+\cos\alpha.\tan\phi)$$

$$= W(\sin\alpha+\mu\cos\alpha) \qquad ...(\because \mu = \tan\phi)$$

2. *Considering the motion of the body down the plane*

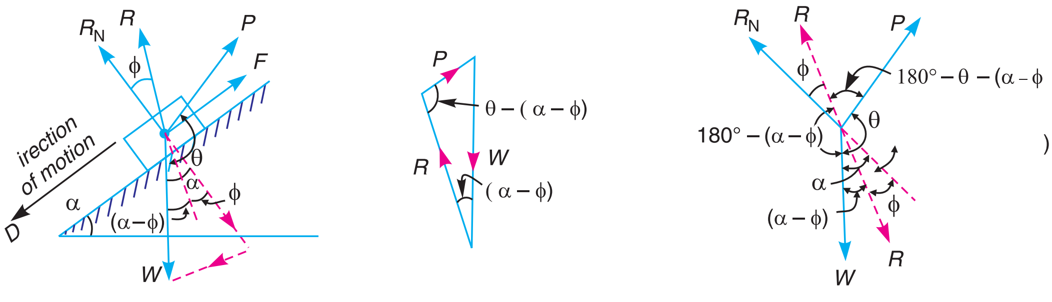


Fig. 10.9. Motion of the body down the plane, considering friction.

When the friction is taken into account, the force of friction $F = \mu.R_N$ will act up the plane and the resultant reaction R will make an angle ϕ with R_N towards its right as shown in Fig. 10.9 (*a*). The triangle of forces is shown in Fig. 10.9 (*b*). Now from sine rule,

$$\frac{P}{\sin(\alpha-\phi)} = \frac{W}{\sin[\theta-(\alpha-\phi)]}$$

or
$$P = \frac{W\sin(\alpha-\phi)}{\sin[\theta-(\alpha-\phi)]} \qquad ...(iv)$$

10.15. Efficiency of Inclined Plane

The ratio of the effort required neglecting friction (*i.e.* P_0) to the effort required considering friction (*i.e.* P) is known as efficiency of the inclined plane. Mathematically efficiency of the inclined plane,

1. *For the motion of the body up the plane*

$$\eta = \frac{\cot(\alpha+\phi)-\cot\theta}{\cot\alpha-\cot\theta}$$

2. *For the motion of the body down the plane*

$$\eta = \frac{\cot\alpha-\cot\theta}{\cot(\alpha-\phi)-\cot\theta}$$

Example 1. *An effort of 1500 N is required to just move a certain body up an inclined plane of angle 12°, force acting parallel to the plane. If the angle of inclination is increased to 15 °, then the effort required is 1720 N. Find the weight of the body and the coefficient of friction.*

Solution. Given : P_1 = 1500 N ; α_1 = 12° ; α_2 = 15° ; P_2 = 1720 N

Let W = Weight of the body in newtons, and μ = Coefficient of friction.

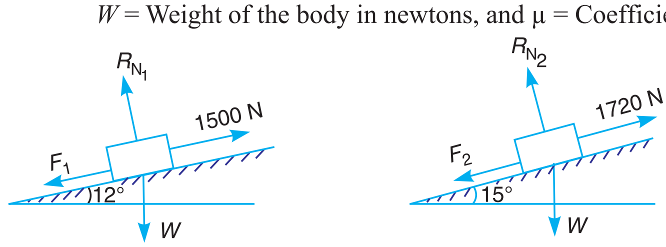


Fig. 10.10

First of all, let us consider a body lying on a plane inclined at an angle of 12º with the horizontal and subjected to an effort of 1500 N parallel to the plane as shown in Fig. 10.10 (*a*).

Let R_{N_1} = Normal reaction, and

F_1 = Force of friction.

We know that for the motion of the body up the inclined plane, the effort applied parallel to the plane (P_1),

$$1500 = W (\sin \alpha_1 + \mu \cos \alpha_1) = W (\sin 12^\circ + \mu \cos 12^\circ) \quad ...(i)$$

Now let us consider the body lying on a plane inclined at an angle of 15º with the horizontal and subjected to an effort of 1720 N parallel to the plane as shown in Fig. 10.10 (*b*).

Let R_{N_2} = Normal reaction, and

F_2 = Force of friction.

We know that for the motion of the body up the inclined plane, the effort applied parallel to the plane (P_2),

$$1720 = W (\sin \alpha_2 + \mu \cos \alpha_2) = W (\sin 15^\circ + \mu \cos 15^\circ) \quad ...(ii)$$

Coefficient of friction

Dividing equation **(*ii*)** by equation **(*i*)**,

$$\frac{1720}{1500} = \frac{W (\sin 15^\circ + \mu \cos 15^\circ)}{W (\sin 12^\circ + \mu \cos 12^\circ)}$$

$$1720 \sin 12^\circ + 1720 \mu \cos 12^\circ = 1500 \sin 15^\circ + 1500 \mu \cos 15^\circ$$

$$\mu (1720 \cos 12^\circ - 1500 \cos 15^\circ) = 1500 \sin 15^\circ - 1720 \sin 12^\circ$$

$$\therefore \mu = \frac{1500 \sin 15^\circ - 1720 \sin 12^\circ}{1720 \cos 12^\circ - 1500 \cos 15^\circ} = \frac{1500 \times 0.2588 - 1720 \times 0.2079}{1720 \times 0.9781 - 1500 \times 0.9659}$$

$$= \frac{388.2 - 357.6}{1682.3 - 1448.5} = \frac{30.6}{233.8} = 0.131 \text{ Ans.}$$

Weight of the body

Substituting the value of μ in equation **(*i*)**,

$$1500 = W (\sin 12^\circ + 0.131 \cos 12^\circ) = W (0.2079 + 0.131 \times 0.9781) = 0.336\, W$$

$$\therefore \quad W = 1500/0.336 = 4464 \text{ N } \textbf{Ans.}$$

10.16. Screw Friction

The screws, bolts, studs, nuts etc. are widely used in various machines and structures for temporary fastenings. The screw threads are mainly of two types *i.e.* V-threads and squarethreads. The V-threads are stronger and ofer more frictional resistance to motion than square threads.

The following terms are important for the study of screw :

1. ***Helix.*** It is the curve traced by a particle while moving along a screw thread.

2. ***Pitch***. It is the distance from a point of a screw to a corresponding point on the next thread, measured parallel to the axis of the screw.

3. ***Lead***. It is the distance, a screw thread advances axially in one turn.

4. ***Depth of thread***. It is the distance between the top and bottom surfaces of a thread (also known as **crest** and **root** of a thread).

5. ***Single-threaded screw***. If the lead of a screw is equal to its pitch, it is known as single threaded screw.

6. ***Multi-threaded screw.*** If more than one thread is cut in one lead distance of a screw, it is known as multi-threaded screw.

Lead = Pitch × Number of threads

7. ***Helix angle.*** It is the slope or inclination of the thread with the horizontal.

$$\tan \alpha = \frac{\text{Lead of screw}}{\text{Circumference of screw}}$$

$= p/\pi d$ ...(In single-threaded screw)

$= n.p/\pi d$ ...(In multi-threaded screw)

where α = Helix angle,

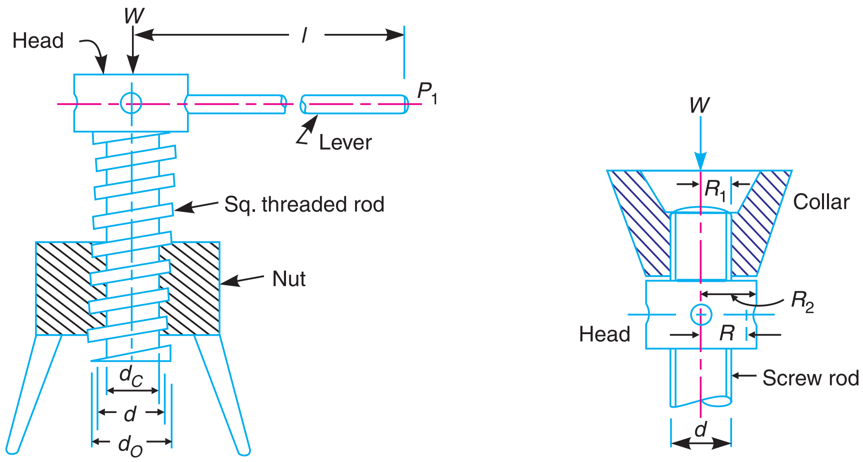
p = Pitch of the screw,

d = Mean diameter of the screw, and

n = Number of threads in one lead.

10.17. Screw Jack

The screw jack is a device, for lifting heavy loads, by applying a comparatively smaller effort at its handle. The principle, on which a screw jack works is similar to that of an inclined plane.

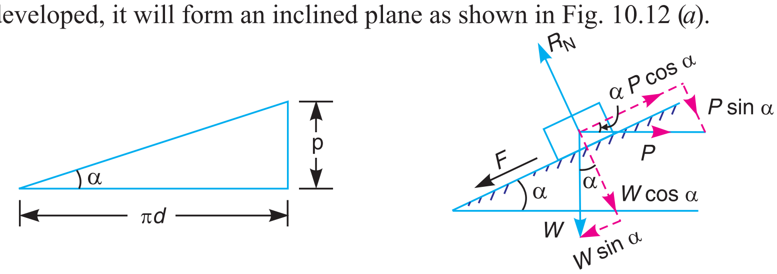


(*a*) Screw jack. (*b*) Thrust collar.

Fig. 10.11

10.18. Torque Required to Lift the Load by a Screw Jack

If one complete turn of a screw thread by imagined to be unwound, from the body of the screw and developed, it will form an inclined plane as shown in Fig. 10.12 (*a*).



(*a*) Development of a screw. (*b*) Forces acting on the screw.

Fig. 10.12

Let p = Pitch of the screw,

d = Mean diameter of the screw,

α = Helix angle,

P = Effort applied at the circumference of the screw to lift the load,

W = Load to be lifted, and

μ = Coefficient of friction, between the screw and nut = $\tan \phi$, where ϕ is the friction angle.

From the geometry of the Fig. 10.12 (*a*), we find that

$$\tan \alpha = p/\pi d$$

NOTE: Lift the load by screw jack is same as raise a load by inclined plane under applied load is horizontal.

$$\frac{P}{\sin(\alpha+\phi)} = \frac{W}{\sin[\theta-(\alpha+\phi)]}$$

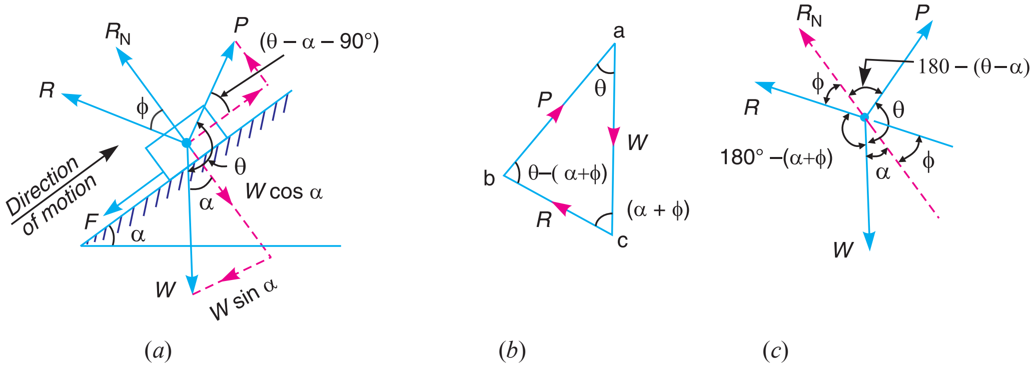


Fig. 10.8. Motion of the body up the plane, considering friction.

$$\therefore \qquad P = \frac{W\sin(\alpha+\phi)}{\sin[\theta-(\alpha+\phi)]} \qquad \ldots(ii)$$

Notes : 1. When the effort applied is horizontal, then $\theta = 90°$. In that case, the equations *(i)* and *(ii)* may be written as

$$P = \frac{W\sin(\alpha+\phi)}{\sin[90° - (\alpha+\phi)]} = \frac{W\sin(\alpha+\phi)}{\cos(\alpha+\phi)} = W\tan(\alpha+\phi)$$

$\therefore$ Torque required to overcome friction between the screw and nut,

$$T_1 = P \times \frac{d}{2} = W\ \tan(\alpha+\phi)\frac{d}{2}$$

When the axial load is taken up by a thrust collar or a flat surface, as shown in Fig. 1011(*b*), so that the load does not rotate with the screw, then the torque required to overcome friction at the collar,

$$T_2 = \mu_1.W\left(\frac{R_1+R_2}{2}\right) = \mu_1.W.R$$

where R_1 and R_2 = Outside and inside radii of the collar,

R = Mean radius of the collar, and

μ_1 = Coefficient of friction for the collar.

$\therefore$ Total torque required to overcome friction (*i.e.* to rotate the screw),

$$T = T_1 + T_2 = P \times \frac{d}{2} + \mu_1.W.R$$

* The **nominal diameter** of a screw thread is also known as **outside diameter** or **major diameter.**

** The **core diameter** of a screw thread is also known as inner diameter or **root diameter** or **minor diameter.**

10.19. Torque Required to Lower the Load by a Screw Jack

We have discussed in Art. 10.18, that the principle on which the screw jack works is similar to that of an inclined plane. If one complete turn of a screw thread be imagined to be unwound from the body of the screw and developed, it will form an inclined plane as shown in Fig. 10.13 (*a*).

Let p = Pitch of the screw,

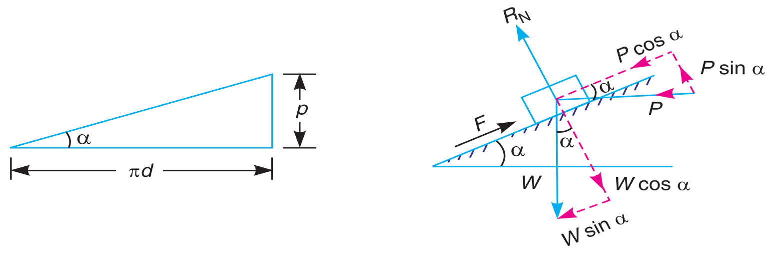
d = Mean diameter of the screw,

α = Helix angle,

P = Effort applied at the circumference of the screw to lower the load,

W = Weight to be lowered, and

μ = Coefficient of friction between the screw and nut = tan ϕ, where ϕ is the friction angle.



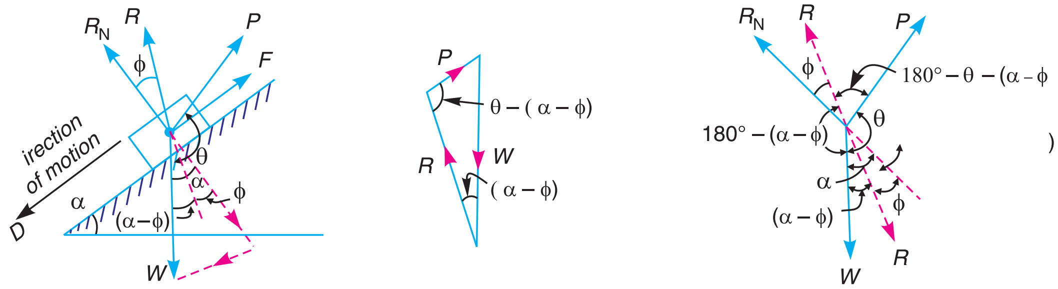
(*a*) (*b*)

Fig. 10.13

From the geometry of the figure, we find that

$$\tan\alpha = p/\pi d$$

NOTE: Lower the load by screw jack is same as Lower a load by inclined plane under applied load is horizontal.



(*a*) (*b*) (*c*)

Fig. 10.9. Motion of the body down the plane, considering friction.

When the friction is taken into account, the force of friction $F = \mu.R_N$ will act up the plane and the resultant reaction R will make an angle ϕ with R_N towards its right as shown in Fig. 10.9 (*a*). The triangle of forces is shown in Fig. 10.9 (*b*). Now from sine rule,

$$\frac{P}{\sin(\alpha-\phi)} = \frac{W}{\sin[\theta-(\alpha-\phi)]}$$

$$P = \frac{W\sin(\alpha\,\delta\,\phi)}{\sin[\theta-(\alpha\,\delta\phi)]} \qquad \text{...}(ii)$$

Notes : 1. When the effort applied is horizontal, then $\theta = 90°$. In that case, the equations (*i*) and (*ii*) may be written as

$$P = \frac{W\sin(\alpha+\phi)}{\sin[90°-(\alpha+\phi)]} = \frac{W\sin(\alpha+\phi)}{\cos(\alpha+\phi)} = W\tan(\alpha+\phi)$$

∴ Torque required to overcome friction between the screw and nut,

$$T = P\times\frac{d}{2} = W\tan(\phi-\alpha)\,\frac{d}{2}$$

Note : When $\alpha > \phi$, then $P = \tan(\alpha - \phi)$.

Example 2. *The mean diameter of a square threaded screw jack is 50 mm. The pitch of the thread is 10 mm. The coefficient of friction is 0.15. What force must be applied at the end of a 0.7 m long lever, which is perpendicular to the longitudinal axis of the screw to raise a load of 20 kN and to lower it?*

Solution. Given : $d = 50$ mm $= 0.05$ m ; $p = 10$ mm ; $\mu = \tan\phi = 0.15$; $l = 0.7$ m ; $W = 20$ kN $= 20 \times 10^3$ N

We know that $\tan\alpha = \frac{p}{\pi d} = \frac{10}{\pi\times 50} = 0.0637$

Let P_1 = Force required at the end of the lever.

Force required to raise the load

We know that force required at the circumference of the screw,

$$\boxed{P = W\tan(\alpha+\phi) = W\left[\frac{\tan\alpha+\tan\phi}{1-\tan\alpha.\tan\phi}\right]}$$

$$= 20 \times 10^3 \left[\frac{0.0637 + 0.15}{1 - 0.0637 \times 0.15} \right] = 4314 \text{ N}$$

Now the force required at the end of the lever may be found out by the relation,

$$P_1 \times l = P \times d/2$$

$$\therefore \qquad P_1 = \frac{P \times d}{2\,l} = \frac{4314 \times 0.05}{2 \times 0.7} = 154 \text{ N} \quad \textbf{Ans.}$$

Force required to lower the load

We know that the force required at the circumference of the screw,

$$P = W \tan(\phi - \alpha) = W \left[\frac{\tan\phi - \tan\alpha}{1 + \tan\phi . \tan\alpha} \right]$$

$$= 20 \times 10^3 \left[\frac{0.15 - 0.0637}{1 + 0.15 \times 0.0637} \right] = 1710 \text{ N}$$

Now the force required at the end of the lever may be found out by the relation,

$$P_1 \times l = P \times \frac{d}{2} \quad \text{or} \quad P_1 = \frac{P \times d}{2l} = \frac{1710 \times 0.05}{2 \times 0.7} = 61 \text{ N} \quad \textbf{Ans.}$$

10.20. Efficiency of a Screw Jack

The efficiency of a screw jack may be defined as **the ratio between the ideal effort** (*i.e.* the effort required to move the load, neglecting friction) to **the actual effort** (*i.e.* the effort required to move the load taking friction into account).

We

$$\therefore \quad \text{Efficiency,} \quad \eta = \frac{\text{Ideal effort}}{\text{Actual effort}} = \frac{P_0}{P} = \frac{W \tan\alpha}{W \tan(\alpha + \phi)} = \frac{\tan\alpha}{\tan(\alpha + \phi)}$$

which shows that the efficiency of a screw jack, is independent of the load raised.

Example 3. *The pitch of 50 mm mean diameter threaded screw of a screw jack is 12.5 mm. The coefficient of friction between the screw and the nut is 0.13. Determine the torque required on the screw to raise a load of 25 kN, assuming the load to rotate with the screw. Determine the ratio of the torque required to raise the load to the torque required to lower the load and also the efficiency of the machine.*

Solution. Given : $d = 50$ mm ; $p = 12.5$ mm ; $\mu = \tan\phi = 0.13$; $W = 25$ kN $= 25 \times 10^3$ N

We know that,
$$\tan\alpha = \frac{p}{\pi\, d} = \frac{12.5}{\pi \times 50} = 0.08$$

and force required on the screw to raise the load,

$$P = W \tan(\alpha + \phi) = W \left[\frac{\tan\phi - \tan\alpha}{1 + \tan\phi . \tan\alpha} \right]$$

$$= 25 \times 10^3 \left[\frac{0.08 + 0.13}{1 - 0.08 \times 0.13} \right] = 5305 \text{ N}$$

Torque required on the screw

We know that the torque required on the screw to raise the load,

$$T_1 = P \times d/2 = 5305 \times 50/2 = 132\,625 \text{ N-mm} \quad \textbf{Ans.}$$

Ratio of the torques required to raise and lower the load

We know that the force required on the screw to lower the load,

$$P = W \tan(\phi - \alpha) = W \left[\frac{\tan\phi - \tan\alpha}{1 + \tan\phi . \tan\alpha} \right]$$

$$= 25 \times 10^3 \left[\frac{0.13 + 0.08}{1 + 0.13 \times 0.08} \right] = 1237 \text{ N}$$

and torque required to lower the load

$$T_2 = P \times d/2 = 1237 \times 50/2 = 30\,905 \text{ N-mm}$$

$\therefore$ Ratio of the torques required,

$$= T_1 / T_2 = 132625/30925 = 4.3 \quad \textbf{Ans.}$$

Efficiency of the machine, We know that the efficiency,

$$\eta = \frac{\tan\alpha}{\tan(\alpha+\phi)} = \frac{\tan\alpha(1-\tan\alpha.\tan\phi)}{\tan\alpha+\tan\phi} = \frac{0.08(1-0.08\times 0.13)}{0.08+0.13}$$

$$\because = 0.377 = 37.7\% \quad \textbf{Ans}$$

10.22. Over Hauling and Self Locking Screws

10.24. Friction of a V-thread

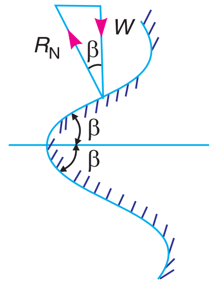


Fig. 10.14. V-thread.

Let 2β = Angle of the V-thread, and

β = Semi-angle of the V-thread.

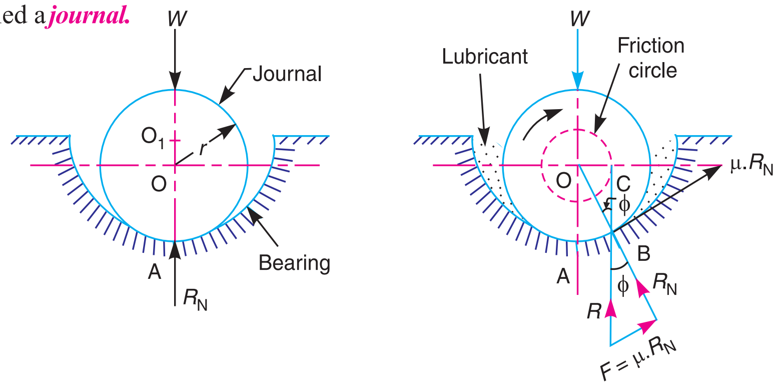
$$\therefore \quad R_N = \frac{W}{\cos\beta}$$

and frictional force, $F = \mu.R_N = \mu \times \dfrac{W}{\cos\beta} = \mu_1.W$

where $\dfrac{\mu}{\cos\beta} = \mu_1$, known as virtual coefficient of friction.

10.25. Friction in Journal Bearing-Friction Circle

A journal bearing forms a turning pair as shown in Fig. 10.15 (*a*). The fixed outer element of a turning pair is called a ***bearing*** and that portion of the inner element (*i.e.* shaft) which fits in the bearing is called a ***journal.***



(*a*) (*b*)

Fig. 10.15. Friction in journal bearing.

When the bearing is not lubricated (or the journal is stationary), then there is a line contact between the two elements as shown in Fig. 10.15 (*a*).

Now consider a shaft rotating inside a bearing in clockwise direction as shown in Fig. 10.15 (*b*). The lubricant between the journal and bearing forms a thin layer which gives rise to a greasy friction. Therefore, the reaction R does not act vertically upward, but acts at another point of pressure B.

In order that the rotation may be maintained, there must be a couple rotating the shaft.

Let ϕ = Angle between R (resultant of F and R_N) and R_N,

μ = Coefficient of friction between the journal and bearing,

T = Frictional torque in N-m, and

r = Radius of the shaft in metres.

For uniform motion, the resultant force acting on the shaft must be zero and the resultant turning moment on the shaft must be zero. In other words,

$$R = W, \text{ and } T = W \times OC = W \times OB \sin\phi = W.r\sin\phi$$

Since ϕ is very small, therefore substituting $\sin\phi = \tan\phi$

$$\therefore \quad T = W.r\tan\phi = \mu.W.r \qquad \ldots(\because\ \mu = \tan\phi)$$

If the shaft rotates with angular velocity ω rad/s, then power wasted in friction,

$$P = T.\omega = T \times 2\pi N/60 \text{ watts}$$

where N = Speed of the shaft in r.p.m.

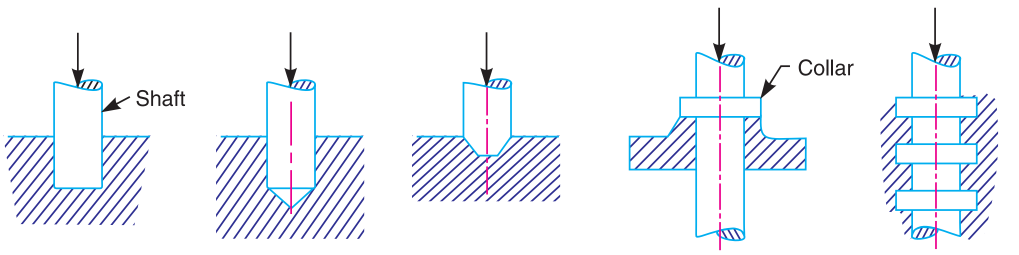
Notes : **1.** If a circle is drawn with centre O and radius $OC = r\sin\phi$, then this circle is called the ***friction circle*** of a bearing.

10.26. Friction of Pivot and Collar Bearing

10.26. Friction of Pivot and Collar Bearing

The rotating shafts are frequently subjected to axial thrust. The bearing surfaces such as pivot and collar bearings are used to take this axial thrust of the rotating shaft.

The bearing surfaces placed at the end of a shaft to take the axial thrust are known as ***pivots.***



(*a*) Flat pivot. (*b*) Conical pivot. (*c*) Truncated pivot. (*d*) Single flat collar. (*e*) Multiple flat collar.

Fig. 10.16. Pivot and collar bearings.

Hence, in the study of friction of bearings, it is assumed that

1. The pressure is uniformly distributed throughout the bearing surface, and
2. The wear is uniform throughout the bearing surface.

10.27. Flat Pivot Bearing

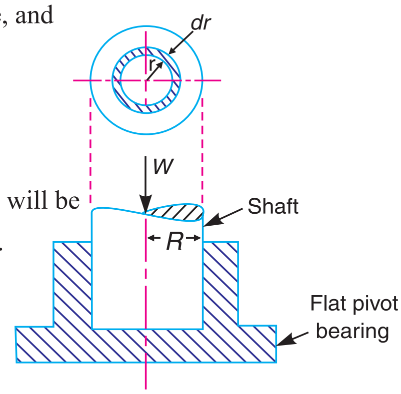


Fig. 10.17. Flat pivot or footstep bearing.

When a vertical shaft rotates in a flat pivot bearing (known as **foot step bearing**), as shown in Fig. 10.17, the sliding friction will be along the surface of contact between the shaft and the bearing.

Let W = Load transmitted over the bearing surface,

R = Radius of bearing surface,

p = Intensity of pressure per unit area of bearing surface between rubbing surfaces, and

μ = Coefficient of friction.

We will consider the following two cases :

1. Considering uniform pressure:

When the pressure is uniformly distributed over the bearing area, then

$$p = \frac{W}{\pi R^2}$$

Consider a ring of radius r and thickness dr of the bearing area.

$\therefore$ Area of bearing surface, $A = 2\pi r.dr$

Load transmitted to the ring,

$$\delta W = p \times A = p \times 2\,\pi\, r.dr \qquad \text{...}(i)$$

Frictional resistance to sliding on the ring acting tangentially at radius r,

$$F_r = \mu.\delta W = \mu\, p \times 2\pi\, r.dr = 2\pi\, \mu.p.r.dr$$

$\therefore$ Frictional torque on the ring,

$$T_r = F_r \times r = 2\pi\, \mu\, p\, r.dr \times r = 2\,\pi\,\mu\, p\, r^2\, dr \qquad \text{...}(ii)$$

Integrating this equation within the limits from 0 to R for the total frictional torque on the pivot bearing.

$$\therefore \text{ Total frictional torque, } T = \int_0^R 2\pi\mu p\, r^2 dr = 2\pi\mu p \int_0^R r^2\, dr$$

$$= 2\pi\mu\, p\left[\frac{r^3}{3}\right]_0^R = 2\pi\mu\, p \times \frac{R^3}{3} = \frac{2}{3} \times \pi\mu.p.R^3$$

$$= \frac{2}{3} \times \pi \mu \times \frac{W}{\pi R^2} \times R^3 = \frac{2}{3} \times \mu . W . R \qquad \ldots\left(\because \; p = \frac{W}{\pi R^2}\right)$$

When the shaft rotates at ω rad/s, then power lost in friction,

$$P = T.\omega = T \times 2\pi N/60 \qquad \ldots(\because \; \omega = 2\pi N/60)$$

2. ***Considering uniform wear***

We have already discussed that the rate of wear depends upon the intensity of pressure (p) and the velocity of rubbing surfaces (v). It is assumed that the rate of wear is proportional to the product of intensity of pressure and the velocity of rubbing surfaces (*i.e.* $p.v.$). Since the velocity of rubbing surfaces increases with the distance (*i.e.* radius r) from the axis of the bearing, therefore for uniform wear

$$p.r = C \text{ (a constant)} \quad \text{or} \quad p = C/r$$

and the load transmitted to the ring,

$$\delta W = p \times 2\pi r.dr \qquad \ldots\text{[From equation } (i)]$$

$$= \frac{C}{r} \times 2\pi r.dr = 2\pi C.dr$$

∴ Total load transmitted to the bearing

$$W = \int_0^R 2\pi \, C.dr = 2\pi \, C[r]_0^R = 2\pi C.R \quad \text{or} \quad C = \frac{W}{2\pi R}$$

We know that frictional torque acting on the ring,

$$T_r = 2\pi \mu \, p r^2 \, dr = 2\pi\mu \times \frac{C}{r} \times r^2 \, dr \qquad \ldots\left(\because \; p = \frac{C}{r}\right)$$

$$= 2\pi \, \mu.C.r \, dr \qquad \ldots(iii)$$

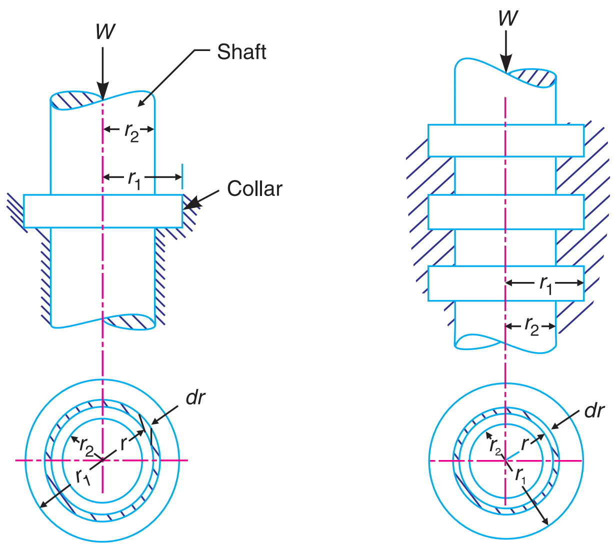
∴ Total frictional torque on the bearing,

$$T = \int_0^R 2\pi \, \mu.C.r.dr = 2\pi\mu.C\left[\frac{r^2}{2}\right]_0^R = 2\pi\mu.C \times \frac{R^2}{2} = \pi\mu.C.R^2$$

$$= \pi\mu \times \frac{W}{2\pi R} \times R^2 = \frac{1}{2} \times \mu.W.R$$

10.30. Flat Collar Bearing

We have already discussed that collar bearings are used to take the axial thrust of the rotating shafts. There may be a single collar or multiple collar bearings as shown in Fig. 10.20 (*a*) and (*b*) respectively. The collar bearings are also known as ***thrust bearings.***



(*a*) Single collar bearing (*b*) Multiple collar bearing.

Fig. 10.20. Flat collar bearings.

1. *Considering uniform pressure*

When the pressure is uniformly distributed over the bearing surface, then the intensity of pressure,

$$p = \frac{W}{A} = \frac{W}{\pi[(r_1)^2 - (r_2)^2]} \qquad \ldots(i)$$

We have seen in Art. 10.25, that the frictional torque on the ring of radius r and thickness dr,

$$T_r = 2\,\pi\mu.p.r^2.dr$$

Integrating this equation within the limits from r_2 to r_1 for the total frictional torque on the collar.

$\therefore$ Total frictional torque,

$$T = \int_{r_2}^{r_1} 2\,\pi\mu.p.r^2.dr = 2\pi\mu.p\left[\frac{r^3}{3}\right]_{r_2}^{r_1} = 2\pi\mu.p\left[\frac{(r_1)^3 - (r_2)^3}{3}\right]$$

Substituting the value of p from equation (*i*),

$$T = 2\pi\mu \times \frac{W}{\pi[(r_1)^2 - (r_2)^2]}\left[\frac{(r_1)^3 - (r_2)^3}{3}\right]$$

$$= \frac{2}{3} \times \mu.W\left[\frac{(r_1)^3 - (r_2)^3}{(r_1)^2 - (r_2)^2}\right]$$

2. *Considering unifrom wear*

We have seen in Art. 10.25 that the load transmitted on the ring, considering uniform wear is,

$$\delta W = p_r.2\pi r.dr = \frac{C}{r} \times 2\pi r.dr = 2\pi C.dr$$

$\therefore$ Total load transmitted to the collar,

$$W = \int_{r_2}^{r_1} 2\pi\,C.dr = 2\pi\,C[r]_{r_2}^{r_1} = 2\pi\,C(r_1 - r_2)$$

We also know that frictional torque on the ring,

$$T_r = \mu.\delta W.r = \mu \times 2\pi C.dr.r = 2\pi\mu.C.r.dr$$

$\therefore$ Total frictional torque on the bearing,

$$T = \int_{r_2}^{r_1} 2\pi\mu\,C.r.dr = 2\pi\mu.C\left[\frac{r^2}{2}\right]_{r_2}^{r_1} = 2\pi\mu.C\left[\frac{(r_1)^2 - (r_2)^2}{2}\right]$$

$$= \pi\mu.C[(r_1)^2 - (r_2)^2]$$

Substituting the value of C from equation (*ii*),

$$T = \pi\mu \times \frac{W}{2\pi(r_1 - r_2)}[(r_1)^2 - (r_2)^2] = \frac{1}{2} \times \mu.W(r_1 + r_2)$$

FRICTION:

when ever one block moves or tends to move tangentially with respect to the surface, on which it rests, the interlocking property of the projecting particles opposes the motion. This opposing force, which acts in the opposite direction of the movement of the upper block, is called the force of friction or simply friction.

Types of Friction: In general, the friction is of the following two types :

1. *Static friction.* It is the friction, experienced by a body, when at rest.

2. *Dynamic friction.* It is the friction, experienced by a body, when in motion. The dynamic friction is also called ***kinetic friction*** and is less than the static friction.

It is of the following three types :

(*a*) ***Sliding friction.*** It is the friction, experienced by a body , when it ***slides*** over another body.

(*b*) ***Rolling friction.*** It is the friction, experienced between the surfaces which has ***balls*** or ***rollers*** interposed between them.

(*c*) ***Pivot friction.*** It is the friction, experienced by a body, due to the ***motion of rotation*** as in case of foot step bearings.

The friction may further be classified as :

10.3. Friction Between Unlubricated Surfaces

The friction experienced between two dry and unlubricated surfaces in contact is known as ***dry*** or ***solid friction***.

10.4. Friction Between Lubricated Surfaces

When lubricant (*i.e.* oil or grease) is applied between two surfaces in contact, then the friction may be classified into the following two types depending upon the thickness of layer of a lubricant.

1. *Boundary friction (or greasy friction or non-viscous friction).* It is the friction, experienced between the rubbing surfaces, when the surfaces have a very thin layer of lubricant.

2. *Fluid friction (or film friction or viscous friction).* It is the friction, experienced between the rubbing surfaces, when the surfaces have a thick layer of the lubrhicant.

10.5. Limiting Friction

The maximum value of frictional force, which comes into play, when a body just begins to slide over the surface of the other body, is known as limiting force of friction or simply limiting friction.

10.6. Laws of Static Friction

Following are the laws of static friction :

1. The force of friction always acts in a direction, opposite to that in which the body tends to move.
2. The magnitude of the force of friction is exactly equal to the force, which tends the body to move.
3. The magnitude of the limiting friction (F) bears a constant ratio to the normal reaction (R_N) between the two surfaces. Mathematically

$$F/R_N = \text{constant}$$

4. The force of friction is independent of the area of contact, between the two surfaces.

5.The force of friction depends upon the roughness of the surfaces.

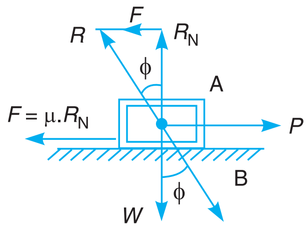
COEFFICIENT OF FRICTION: It is defined as the ratio of the limiting friction (F) to the normal reaction (RN) between the two bodies.

$$\mu = F/RN$$

ANGLE OF FRICTION:

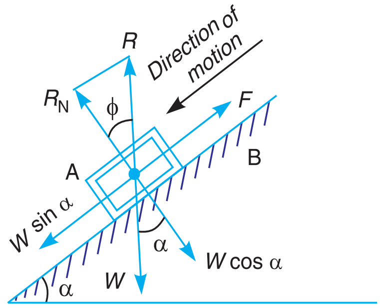
It may be defined as the angle which the resultant reaction R makes with the normal reaction RN.

tan φ = F/RN = μRN/RN = μ



ANGLE OF REPOSE:

If the angle of inclination α of the plane to the horizontal is such that the body begins to move down the plane, then the angle α is called the angle of repose. A little consideration will show that the body will begin to move down the plane when the angle of inclination of the plane is equal to the angle of friction (i.e. α = φ).



10.14. Friction of a Body Lying on a Rough Inclined Plane

1. *Considering the motion of the body up the plane*

Let W = Weight of the body,

α = Angle of inclination of the plane to the horizontal,

ϕ = Limiting angle of friction for the contact surfaces,

P = Effort applied in a given direction in order to cause the body to slide with uniform velocity parallel to the plane, considering friction,

P_0 = Effort required to move the body up the plane neglecting friction,

θ = Angle which the line of action of P makes with the weight of the body W,

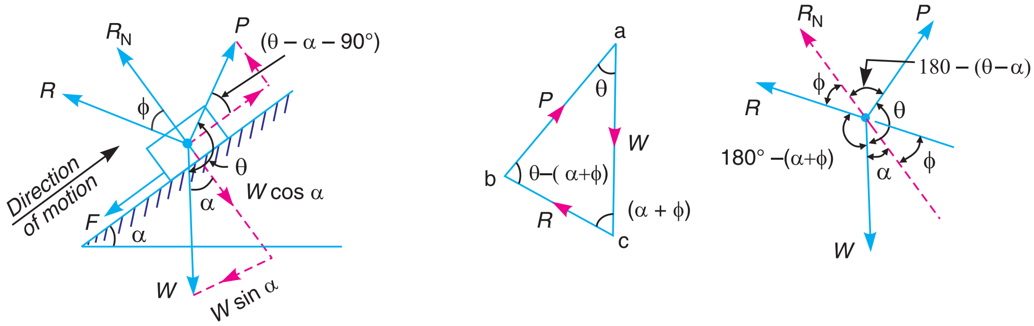
μ = Coefficient of friction between the surfaces of the plane and the body,

R_N = Normal reaction, and

R = Resultant reaction.

When friction is taken into account, a frictional force $F = \mu.R_N$ acts in the direction opposite to the motion of the body, as shown in Fig. 10.8 (*a*). The resultant reaction R between the plane and the body is inclined at an angle ϕ with the normal reaction R_N. The triangle of forces is shown in Fig. 10.8 (*b*). Now applying sine rule,

$$\frac{P}{\sin(\alpha + \phi)} = \frac{W}{\sin[\theta - (\alpha + \phi)]}$$



(*a*) (*b*) (*c*)

Fig. 10.8. Motion of the body up the plane, considering friction.

$$\therefore \qquad P = \frac{W \sin(\alpha + \phi)}{\sin[\theta - (\alpha + \phi)]} \qquad \text{...}(ii)$$

Notes : 1. When the effort applied is horizontal, then θ = 90°. In that case, the equations **(*i*)** and **(*ii*)** may be written as

$$P = \frac{W \sin(\alpha + \phi)}{\sin[90^\circ - (\alpha + \phi)]} = \frac{W \sin(\alpha + \phi)}{\cos(\alpha + \phi)} = W \tan(\alpha + \phi)$$

2. When the effort applied is parallel to the plane, then $\theta = 90^\circ + \alpha$. In that case, the equations ***(i)*** and ***(ii)*** may be written as

$$P = \frac{W \sin(\alpha + \phi)}{\sin[(90^\circ + \alpha) - (\alpha + \phi)]} = \frac{W \sin(\alpha + \phi)}{\cos \phi}$$

$$= \frac{W(\sin \alpha \cos \phi + \cos \alpha \sin \phi)}{\cos \phi} = W(\sin \alpha + \cos \alpha . \tan \phi)$$

$$= W(\sin \alpha + \mu \cos \alpha) \qquad ...(\because \mu = \tan \phi)$$

2. Considering the motion of the body down the plane

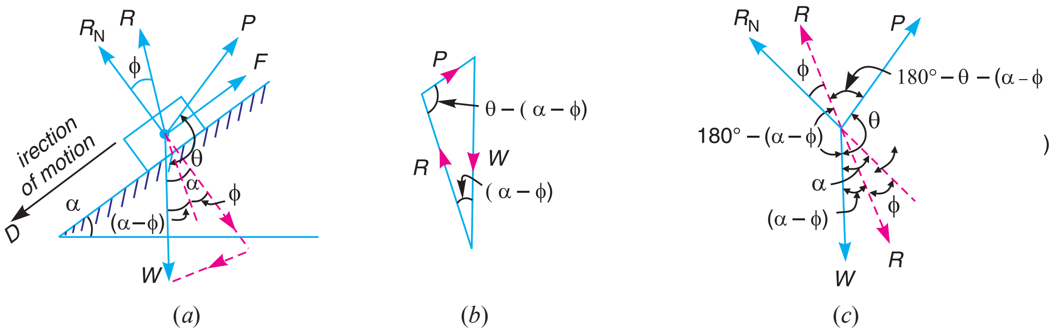


Fig. 10.9. Motion of the body down the plane, considering friction.

When the friction is taken into account, the force of friction $F = \mu . R_N$ will act up the plane and the resultant reaction R will make an angle ϕ with R_N towards its right as shown in Fig. 10.9 (*a*). The triangle of forces is shown in Fig. 10.9 (*b*). Now from sine rule,

$$\frac{P}{\sin(\alpha - \phi)} = \frac{W}{\sin[\theta - (\alpha - \phi)]}$$

or
$$P = \frac{W \sin(\alpha - \phi)}{\sin[\theta - (\alpha - \phi)]} \qquad ...(iv)$$

10.15. Efficiency of Inclined Plane

The ratio of the effort required neglecting friction (*i.e.* P_0) to the effort required considering friction (*i.e.* P) is known as efficiency of the inclined plane. Mathematically efficiency of the inclined plane,

1. For the motion of the body up the plane

$$\eta = \frac{\cot(\alpha + \phi) - \cot \theta}{\cot \alpha - \cot \theta}$$

2. For the motion of the body down the plane

$$\eta = \frac{\cot \alpha - \cot \theta}{\cot(\alpha - \phi) - \cot \theta}$$

Example 1. *An effort of 1500 N is required to just move a certain body up an inclined plane of angle 12°, force acting parallel to the plane. If the angle of inclination is increased to 15°, then the effort required is 1720 N. Find the weight of the body and the coefficient of friction.*

Solution. Given : $P_1 = 1500$ N ; $\alpha_1 = 12^\circ$; $\alpha_2 = 15^\circ$; $P_2 = 1720$ N

Let W = Weight of the body in newtons, and μ = Coefficient of friction.

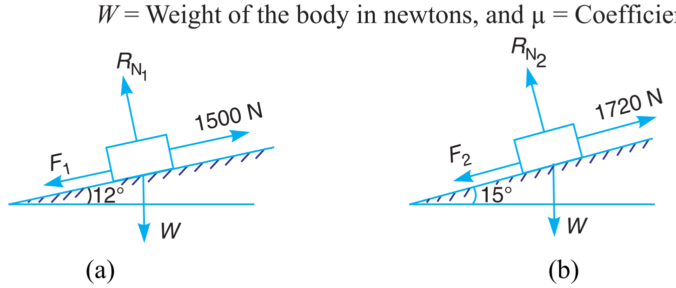


Fig. 10.10

First of all, let us consider a body lying on a plane inclined at an angle of 12º with the horizontal and subjected to an effort of 1500 N parallel to the plane as shown in Fig. 10.10 (*a*).

Let R_{N_1} = Normal reaction, and

F_1 = Force of friction.

We know that for the motion of the body up the inclined plane, the effort applied parallel to the plane (P_1),

$$1500 = W(\sin \alpha_1 + \mu \cos \alpha_1) = W(\sin 12^\circ + \mu \cos 12^\circ) \quad ...(i)$$

Now let us consider the body lying on a plane inclined at an angle of 15º with the horizontal and subjected to an effort of 1720 N parallel to the plane as shown in Fig. 10.10 (*b*).

Let R_{N_2} = Normal reaction, and

F_2 = Force of friction.

We know that for the motion of the body up the inclined plane, the effort applied parallel to the plane (P_2),

$$1720 = W(\sin \alpha_2 + \mu \cos \alpha_2) = W(\sin 15^\circ + \mu \cos 15^\circ) \quad ...(ii)$$

Coefficient of friction

Dividing equation ***(ii)*** by equation ***(i)***,

$$\frac{1720}{1500} = \frac{W(\sin 15^\circ + \mu \cos 15^\circ)}{W(\sin 12^\circ + \mu \cos 12^\circ)}$$

$$1720 \sin 12^\circ + 1720\, \mu \cos 12^\circ = 1500 \sin 15^\circ + 1500\, \mu \cos 15^\circ$$

$$\mu (1720 \cos 12^\circ - 1500 \cos 15^\circ) = 1500 \sin 15^\circ - 1720 \sin 12^\circ$$

$$\therefore \mu = \frac{1500 \sin 15^\circ - 1720 \sin 12^\circ}{1720 \cos 12^\circ - 1500 \cos 15^\circ} = \frac{1500 \times 0.2588 - 1720 \times 0.2079}{1720 \times 0.9781 - 1500 \times 0.9659}$$

$$= \frac{388.2 - 357.6}{1682.3 - 1448.5} = \frac{30.6}{233.8} = 0.131 \textbf{ Ans.}$$

Weight of the body

Substituting the value of μ in equation ***(i)***,

$$1500 = W(\sin 12^\circ + 0.131 \cos 12^\circ) = W(0.2079 + 0.131 \times 0.9781) = 0.336\, W$$

$$\therefore \quad W = 1500/0.336 = 4464 \text{ N } \textbf{Ans.}$$

10.16. Screw Friction

The screws, bolts, studs, nuts etc. are widely used in various machines and structures for temporary fastenings. The screw threads are mainly of two types *i.e.* V-threads and squarethreads. The V-threads are stronger and ofer more frictional resistance to motion than square threads.

The following terms are important for the study of screw :

1. ***Helix.*** It is the curve traced by a particle while moving along a screw thread.

2. ***Pitch***. It is the distance from a point of a screw to a corresponding point on the next thread, measured parallel to the axis of the screw.

3. ***Lead***. It is the distance, a screw thread advances axially in one turn.

4. ***Depth of thread***. It is the distance between the top and bottom surfaces of a thread (also known as **crest** and **root** of a thread).

5. ***Single-threaded screw***. If the lead of a screw is equal to its pitch, it is known as single threaded screw.

6. ***Multi-threaded screw.*** If more than one thread is cut in one lead distance of a screw, it is known as multi-threaded screw.

Lead = Pitch × Number of threads

7. ***Helix angle.*** It is the slope or inclination of the thread with the horizontal.

$$\tan \alpha = \frac{\text{Lead of screw}}{\text{Circumference of screw}}$$

$$= p/\pi d \qquad \text{...(In single-threaded screw)}$$

$$= n.p/\pi d \qquad \text{...(In multi-threaded screw)}$$

where α = Helix angle,

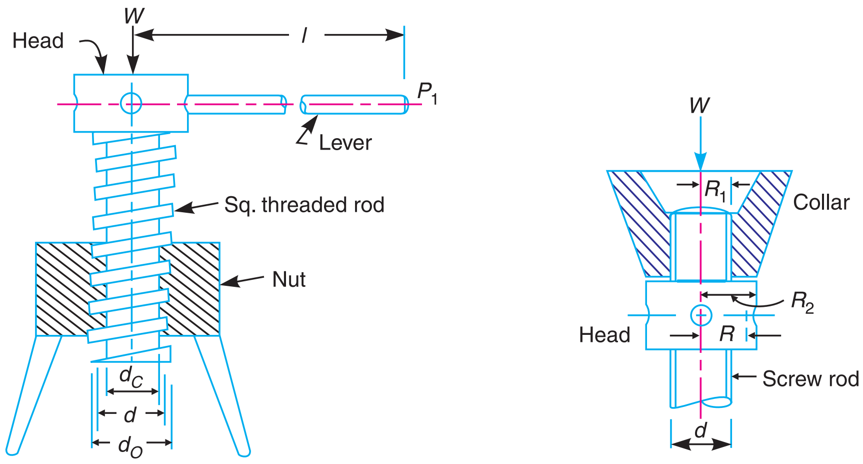
p = Pitch of the screw,

d = Mean diameter of the screw, and

n = Number of threads in one lead.

10.17. Screw Jack

The screw jack is a device, for lifting heavy loads, by applying a comparatively smaller effort at its handle. The principle, on which a screw jack works is similar to that of an inclined plane.

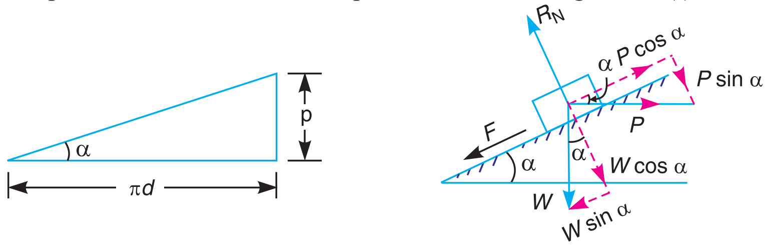


(*a*) Screw jack. (*b*) Thrust collar.

Fig. 10.11

10.18. Torque Required to Lift the Load by a Screw Jack

If one complete turn of a screw thread by imagined to be unwound, from the body of the screw and developed, it will form an inclined plane as shown in Fig. 10.12 (*a*).



(*a*) Development of a screw. (*b*) Forces acting on the screw.

Fig. 10.12

Let p = Pitch of the screw,

d = Mean diameter of the screw,

α = Helix angle,

P = Effort applied at the circumference of the screw to lift the load,

W = Load to be lifted, and

μ = Coefficient of friction, between the screw and nut = $\tan \phi$, where ϕ is the friction angle.

From the geometry of the Fig. 10.12 (*a*), we find that

$$\tan \alpha = p/\pi d$$

Since the principle on which a screw jack works is similar to that of an inclined plane, therefore the force applied on the lever of a screw jack may be considered to be horizontal as shown in Fig. 10.12 (*b*).

Since the load is being lifted, therefore the force of friction ($F = \mu.R_N$) will act downwards. All the forces acting on the screw are shown in Fig. 10.12 (*b*).

Resolving the forces along the plane,

$$P \cos \alpha = W \sin \alpha + F = W \sin \alpha + \mu.R_N \quad ...(i)$$

and resolving the forces perpendicular to the plane,

$$R_N = P \sin \alpha + W \cos \alpha \quad ...(ii)$$

Substituting this value of R_N in equation (*i*),

$$P \cos \alpha = W \sin \alpha + \mu (P \sin \alpha + W \cos \alpha)$$

$$= W \sin \alpha + \mu P \sin \alpha + \mu W \cos \alpha$$

or $$P \cos \alpha - \mu P \sin \alpha = W \sin \alpha + \mu W \cos \alpha$$

or $$P (\cos \alpha - \mu \sin \alpha) = W (\sin \alpha + \mu \cos \alpha)$$

$\therefore$ $$P = W \times \frac{\sin \alpha + \mu \cos \alpha}{\cos \alpha - \mu \sin \alpha}$$

Substituting the value of $\mu = \tan \phi$ in the above equation, we get

$$P = W \times \frac{\sin \alpha + \tan \phi \cos \alpha}{\cos \alpha - \tan \phi \sin \alpha}$$

Multiplying the numerator and denominator by $\cos \phi$,

$$P = W \times \frac{\sin \alpha \cos \phi + \sin \phi \cos \alpha}{\cos \alpha \cos \phi - \sin \alpha \sin \phi} = W \times \frac{\sin (\alpha + \phi)}{\cos (\alpha + \phi)}$$

$$= W \tan (\alpha + \phi)$$

$\therefore$ Torque required to overcome friction between the screw and nut,

$$T_1 = P \times \frac{d}{2} = W \tan (\alpha + \phi) \frac{d}{2}$$

When the axial load is taken up by a thrust collar or a flat surface, as shown in Fig. 10.11(*b*), so that the load does not rotate with the screw, then the torque required to overcome friction at the collar,

$$T_2 = \mu_1.W \left(\frac{R_1 + R_2}{2} \right) = \mu_1.W.R$$

where R_1 and R_2 = Outside and inside radii of the collar,

R = Mean radius of the collar, and

μ_1 = Coefficient of friction for the collar.

$\therefore$ Total torque required to overcome friction (*i.e.* to rotate the screw),

$$T = T_1 + T_2 = P \times \frac{d}{2} + \mu_1.W.R$$

* The **nominal diameter** of a screw thread is also known as **outside diameter** or **major diameter.**

** The **core diameter** of a screw thread is also known as inner diameter or **root diameter** or **minor diameter.**

10.19. Torque Required to Lower the Load by a Screw Jack

We have discussed in Art. 10.18, that the principle on which the screw jack works is similar to that of an inclined plane. If one complete turn of a screw thread be imagined to be unwound from the body of the screw and developed, it will form an inclined plane as shown in Fig. 10.13 (*a*).

Let p = Pitch of the screw,

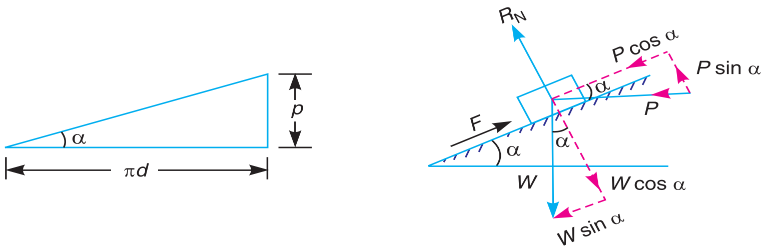
d = Mean diameter of the screw,

α = Helix angle,

P = Effort applied at the circumference of the screw to lower the load,

W = Weight to be lowered, and

μ = Coefficient of friction between the screw and nut = $\tan \phi$, where ϕ is the friction angle.



(*a*) (*b*)

Fig. 10.13

From the geometry of the figure, we find that

$$\tan \alpha = p/\pi d$$

Since the load is being lowered, therefore the force of friction ($F = \mu.R_N$) will act upwards. All the forces acting on the screw are shown in Fig. 10.13 (*b*).

Resolving the forces along the plane,

$$P \cos \alpha = F - W \sin \alpha = \mu.R_N - W \sin \alpha \quad ...(i)$$

and resolving the forces perpendicular to the plane,

$$R_N = W \cos \alpha - P \sin \alpha \quad ...(ii)$$

Substituting this value of R_N in equation (*i*),

$$P \cos \alpha = \mu (W \cos \alpha - P \sin \alpha) - W \sin \alpha$$
$$= \mu.W \cos \alpha - \mu.P \sin \alpha - W \sin \alpha$$

or $$P \cos \alpha + \mu.P \sin \alpha = \mu.W \cos \alpha - W \sin \alpha$$

or $$P (\cos \alpha + \mu \sin \alpha) = W (\mu \cos \alpha - \sin \alpha)$$

$$\therefore \quad P = W \times \frac{(\mu \cos \alpha - \sin \alpha)}{(\cos \alpha + \mu \sin \alpha)}$$

Substituting the value of $\mu = \tan \phi$ in the above equation, we get

$$P = W \times \frac{(\tan \phi \cos \alpha - \sin \alpha)}{(\cos \alpha + \tan \phi \sin \alpha)}$$

Multiplying the numerator and denominator by $\cos \phi$,

$$P = W \times \frac{(\sin \phi \cos \alpha - \sin \alpha \cos \phi)}{(\cos \alpha \cos \phi + \sin \phi \sin \alpha)} = W \times \frac{\sin (\phi - \alpha)}{\cos (\phi - \alpha)}$$

$$= W \tan (\phi - \alpha)$$

$\therefore$ Torque required to overcome friction between the screw and nut,

$$T = P \times \frac{d}{2} = W \tan (\phi - \alpha) \frac{d}{2}$$

Note : When $\alpha > \phi$, then $P = \tan (\alpha - \phi)$.

Example 2. *The mean diameter of a square threaded screw jack is 50 mm. The pitch of the thread is 10 mm. The coefficient of friction is 0.15. What force must be applied at the end of a 0.7 m long lever, which is perpendicular to the longitudinal axis of the screw to raise a load of 20 kN and to lower it?*

Solution. Given : $d = 50$ mm $= 0.05$ m ; $p = 10$ mm ; $\mu = \tan \phi = 0.15$; $l = 0.7$ m ; $W = 20$ kN $= 20 \times 10^3$ N

We know that $$\tan \alpha = \frac{p}{\pi d} = \frac{10}{\pi \times 50} = 0.0637$$

Let P_1 = Force required at the end of the lever.

Force required to raise the load

We know that force required at the circumference of the screw,

$$P = W \tan (\alpha + \phi) = W \left[\frac{\tan \alpha + \tan \phi}{1 - \tan \alpha . \tan \phi} \right]$$

$$= 20 \times 10^3 \left[\frac{0.0637 + 0.15}{1 - 0.0637 \times 0.15} \right] = 4314 \text{ N}$$

Now the force required at the end of the lever may be found out by the relation,

$$P_1 \times l = P \times d/2$$

$$\therefore \qquad P_1 = \frac{P \times d}{2\,l} = \frac{4314 \times 0.05}{2 \times 0.7} = 154 \text{ N} \quad \textbf{Ans.}$$

Force required to lower the load

We know that the force required at the circumference of the screw,

$$P = W \tan(\phi - \alpha) = W \left[\frac{\tan\phi - \tan\alpha}{1 + \tan\phi . \tan\alpha} \right]$$

$$= 20 \times 10^3 \left[\frac{0.15 - 0.0637}{1 + 0.15 \times 0.0637} \right] = 1710 \text{ N}$$

Now the force required at the end of the lever may be found out by the relation,

$$P_1 \times l = P \times \frac{d}{2} \quad \text{or} \quad P_1 = \frac{P \times d}{2l} = \frac{1710 \times 0.05}{2 \times 0.7} = 61 \text{ N} \quad \textbf{Ans.}$$

10.20. Efficiency of a Screw Jack

The efficiency of a screw jack may be defined as **the ratio between the ideal effort** (*i.e.* the effort required to move the load, neglecting friction) to **the actual effort** (*i.e.* the effort required to move the load taking friction into account).

We

$$\therefore \text{ Efficiency, } \eta = \frac{\text{Ideal effort}}{\text{Actual effort}} = \frac{P_0}{P} = \frac{W \tan\alpha}{W \tan(\alpha + \phi)} = \frac{\tan\alpha}{\tan(\alpha + \phi)}$$

which shows that the efficiency of a screw jack, is independent of the load raised.

Example 3. *The pitch of 50 mm mean diameter threaded screw of a screw jack is 12.5 mm. The coefficient of friction between the screw and the nut is 0.13. Determine the torque required on the screw to raise a load of 25 kN, assuming the load to rotate with the screw. Determine the ratio of the torque required to raise the load to the torque required to lower the load and also the efficiency of the machine.*

Solution. Given : $d = 50$ mm ; $p = 12.5$ mm ; $\mu = \tan\phi = 0.13$; $W = 25$ kN $= 25 \times 10^3$ N

We know that, $\qquad \tan\alpha = \dfrac{p}{\pi\, d} = \dfrac{12.5}{\pi \times 50} = 0.08$

and force required on the screw to raise the load,

$$P = W \tan(\alpha + \phi) = W \left[\frac{\tan\phi - \tan\alpha}{1 + \tan\phi . \tan\alpha} \right]$$

$$= 25 \times 10^3 \left[\frac{0.08 + 0.13}{1 - 0.08 \times 0.13} \right] = 5305 \text{ N}$$

Torque required on the screw

We know that the torque required on the screw to raise the load,

$$T_1 = P \times d/2 = 5305 \times 50/2 = 132\,625 \text{ N-mm} \quad \textbf{Ans.}$$

Ratio of the torques required to raise and lower the load

We know that the force required on the screw to lower the load,

$$P = W \tan(\phi - \alpha) = W \left[\frac{\tan\phi - \tan\alpha}{1 + \tan\phi . \tan\alpha} \right]$$

$$= 25 \times 10^3 \left[\frac{0.13 + 0.08}{1 + 0.13 \times 0.08} \right] = 1237 \text{ N}$$

and torque required to lower the load

$$T_2 = P \times d/2 = 1237 \times 50/2 = 30\,905 \text{ N-mm}$$

$\therefore$ Ratio of the torques required,

$$= T_1 / T_2 = 132625 / 30925 = 4.3 \quad \textbf{Ans.}$$

Efficiency of the machine, We know that the efficiency,

$$\eta = \frac{\tan\alpha}{\tan(\alpha+\phi)} = \frac{\tan\alpha(1-\tan\alpha.\tan\phi)}{\tan\alpha+\tan\phi} = \frac{0.08(1-0.08\times 0.13)}{0.08+0.13}$$

$\because = 0.377 = 37.7\%$ **Ans**

10.22. Over Hauling and Self Locking Screws

10.24. Friction of a V-thread

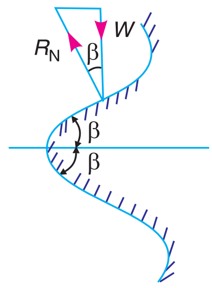


Fig. 10.14. V-thread.

Let 2β = Angle of the V-thread, and

β = Semi-angle of the V-thread.

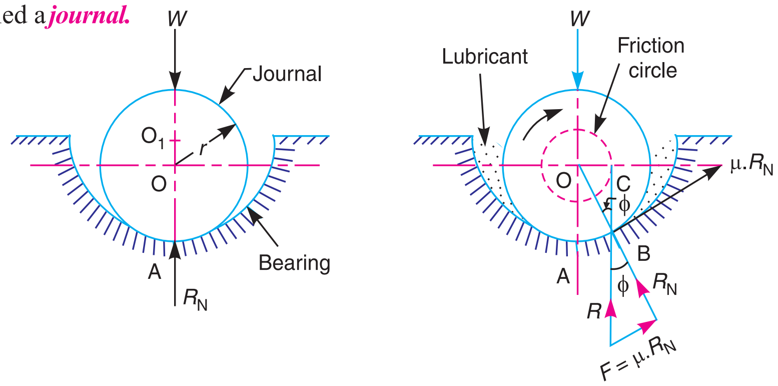
$$\therefore \quad R_N = \frac{W}{\cos\beta}$$

and frictional force, $F = \mu.R_N = \mu \times \frac{W}{\cos\beta} = \mu_1.W$

where $\frac{\mu}{\cos\beta} = \mu_1$, known as virtual coefficient of friction.

10.25. Friction in Journal Bearing-Friction Circle

A journal bearing forms a turning pair as shown in Fig. 10.15 (*a*). The fixed outer element of a turning pair is called a ***bearing*** and that portion of the inner element (*i.e.* shaft) which fits in the bearing is called a ***journal.***



(*a*) (*b*)

Fig. 10.15. Friction in journal bearing.

When the bearing is not lubricated (or the journal is stationary), then there is a line contact between the two elements as shown in Fig. 10.15 (*a*).

Now consider a shaft rotating inside a bearing in clockwise direction as shown in Fig. 10.15 (*b*). The lubricant between the journal and bearing forms a thin layer which gives rise to a greasy friction. Therefore, the reaction R does not act vertically upward, but acts at another point of pressure B.

In order that the rotation may be maintained, there must be a couple rotating the shaft.

Let ϕ = Angle between R (resultant of F and R_N) and R_N,

μ = Coefficient of friction between the journal and bearing,

T = Frictional torque in N-m, and

r = Radius of the shaft in metres.

For uniform motion, the resultant force acting on the shaft must be zero and the resultant turning moment on the shaft must be zero. In other words,

$$R = W, \text{ and } T = W \times OC = W \times OB \sin\phi = W.r\sin\phi$$

Since ϕ is very small, therefore substituting $\sin\phi = \tan\phi$

$$\therefore \quad T = W.r\tan\phi = \mu.W.r \qquad \ldots(\because \mu = \tan\phi)$$

If the shaft rotates with angular velocity ω rad/s, then power wasted in friction,

$$P = T.\omega = T \times 2\pi N/60 \text{ watts}$$

where N = Speed of the shaft in r.p.m.

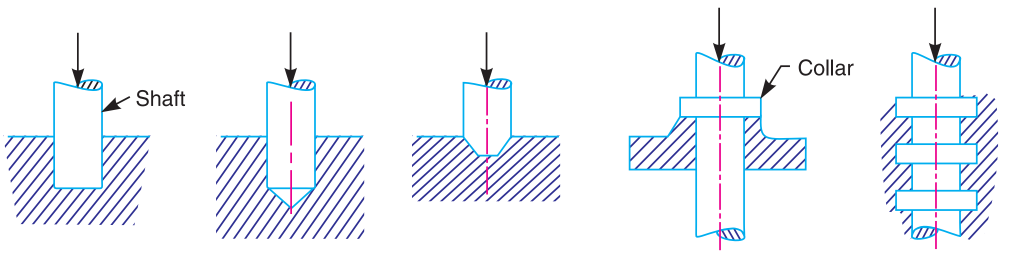
Notes : **1.** If a circle is drawn with centre O and radius $OC = r\sin\phi$, then this circle is called the ***friction circle*** of a bearing.

10.26. Friction of Pivot and Collar Bearing

10.26. Friction of Pivot and Collar Bearing

The rotating shafts are frequently subjected to axial thrust. The bearing surfaces such as pivot and collar bearings are used to take this axial thrust of the rotating shaft.

The bearing surfaces placed at the end of a shaft to take the axial thrust are known as ***pivots.***



(*a*) Flat pivot. (*b*) Conical pivot. (*c*) Truncated pivot. (*d*) Single flat collar. (*e*) Multiple flat collar.

Fig. 10.16. Pivot and collar bearings.

Hence, in the study of friction of bearings, it is assumed that

1. The pressure is uniformly distributed throughout the bearing surface, and
2. The wear is uniform throughout the bearing surface.

10.27. Flat Pivot Bearing

When a vertical shaft rotates in a flat pivot bearing (known as **foot step bearing**), as shown in Fig. 10.17, the sliding friction will be along the surface of contact between the shaft and the bearing.

Let W = Load transmitted over the bearing surface,

R = Radius of bearing surface,

p = Intensity of pressure per unit area of bearing surface between rubbing surfaces, and

μ = Coefficient of friction.

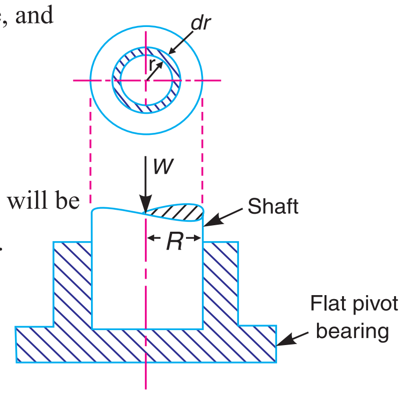


Fig. 10.17. Flat pivot or footstep bearing.

We will consider the following two cases :

1. Considering uniform pressure:

When the pressure is uniformly distributed over the bearing area, then

$$p = \frac{W}{\pi R^2}$$

Consider a ring of radius r and thickness dr of the bearing area.

$\therefore$ Area of bearing surface, $A = 2\pi r.dr$

Load transmitted to the ring,

$$\delta W = p \times A = p \times 2\,\pi\, r.dr \qquad \text{...}(i)$$

Frictional resistance to sliding on the ring acting tangentially at radius r,

$$F_r = \mu.\delta W = \mu\, p \times 2\pi\, r.dr = 2\pi\, \mu.p.r.dr$$

$\therefore$ Frictional torque on the ring,

$$T_r = F_r \times r = 2\pi\, \mu\, p\, r.dr \times r = 2\,\pi\,\mu\, p\, r^2\, dr \qquad \text{...}(ii)$$

Integrating this equation within the limits from 0 to R for the total frictional torque on the pivot bearing.

$$\therefore \text{ Total frictional torque, } T = \int_0^R 2\pi\mu p r^2 dr = 2\pi\mu p \int_0^R r^2\, dr$$

$$= 2\,\pi\mu\, p\left[\frac{r^3}{3}\right]_0^R = 2\,\pi\mu\, p \times \frac{R^3}{3} = \frac{2}{3} \times \pi\mu.p.R^3$$

$$= \frac{2}{3} \times \pi\mu \times \frac{W}{\pi R^2} \times R^3 = \frac{2}{3} \times \mu.W.R \qquad ...\left(\because \ p = \frac{W}{\pi R^2}\right)$$

When the shaft rotates at ω rad/s, then power lost in friction,

$$P = T.\omega = T \times 2\pi N/60 \qquad ...(\because \ \omega = 2\pi N/60)$$

2. *Considering uniform wear*

We have already discussed that the rate of wear depends upon the intensity of pressure (p) and the velocity of rubbing surfaces (v). It is assumed that the rate of wear is proportional to the product of intensity of pressure and the velocity of rubbing surfaces (*i.e.* $p.v.$). Since the velocity of rubbing surfaces increases with the distance (*i.e.* radius r) from the axis of the bearing, therefore for uniform wear

$$p.r = C \text{ (a constant)} \quad \text{or} \quad p = C/r$$

and the load transmitted to the ring,

$$\delta W = p \times 2\pi r.dr \qquad ...[\text{From equation } (i)]$$

$$= \frac{C}{r} \times 2\pi r.dr = 2\pi C.dr$$

∴ Total load transmitted to the bearing

$$W = \int_0^R 2\pi\, C.dr = 2\pi\, C[r]_0^R = 2\pi C.R \quad \text{or} \quad C = \frac{W}{2\pi R}$$

We know that frictional torque acting on the ring,

$$T_r = 2\pi\mu\, p r^2\, dr = 2\pi\mu \times \frac{C}{r} \times r^2\, dr \qquad ...\left(\because \ p = \frac{C}{r}\right)$$

$$= 2\pi\, \mu.C.r\, dr \qquad ...(iii)$$

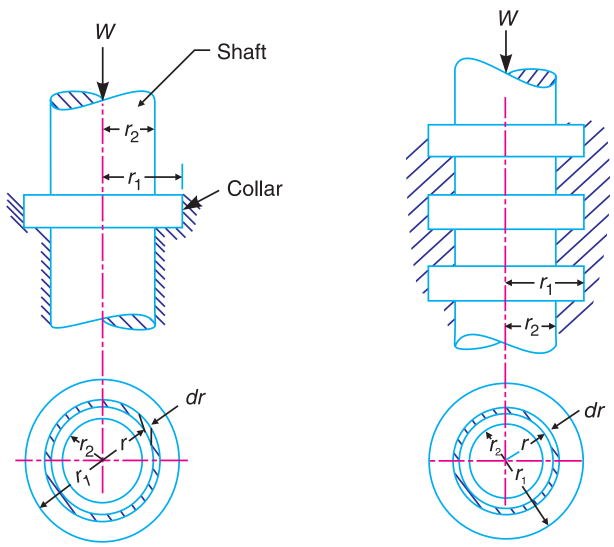
∴ Total frictional torque on the bearing,

$$T = \int_0^R 2\pi\, \mu.C.r.dr = 2\pi\mu.C\left[\frac{r^2}{2}\right]_0^R = 2\pi\mu.C \times \frac{R^2}{2} = \pi\mu.C.R^2$$

$$= \pi\mu \times \frac{W}{2\pi R} \times R^2 = \frac{1}{2} \times \mu.W.R$$

10.30. Flat Collar Bearing

We have already discussed that collar bearings are used to take the axial thrust of the rotating shafts. There may be a single collar or multiple collar bearings as shown in Fig. 10.20 (*a*) and (*b*) respectively. The collar bearings are also known as ***thrust bearings.***



(*a*) Single collar bearing (*b*) Multiple collar bearing.

Fig. 10.20. Flat collar bearings.

1. ***Considering uniform pressure***

When the pressure is uniformly distributed over the bearing surface, then the intensity of pressure,

$$p = \frac{W}{A} = \frac{W}{\pi[r_1)^2 - (r_2)^2]} \qquad \text{...}(i)$$

We have seen in Art. 10.25, that the frictional torque on the ring of radius r and thickness dr,

$$T_r = 2\pi\mu.p.r^2.dr$$

Integrating this equation within the limits from r_2 to r_1 for the total frictional torque on the collar.

$\therefore$ Total frictional torque,

$$T = \int_{r_2}^{r_1} 2\pi\mu.p.r^2.dr = 2\pi\mu.p\left[\frac{r_3}{3}\right]_{r_2}^{r_1} = 2\pi\mu.p\left[\frac{(r_1)^3 - (r_2)^3}{3}\right]$$

Substituting the value of p from equation (*i*),

$$T = 2\pi\mu \times \frac{W}{\pi[(r_1)^2 - (r_2)^2]}\left[\frac{(r_1)^3 - (r_2)^3}{3}\right]$$

$$= \frac{2}{3} \times \mu.W\left[\frac{(r_1)^3 - (r_2)^3}{(r_1)^2 - (r_2)^2}\right]$$

2. ***Considering unifrom wear***

We have seen in Art. 10.25 that the load transmitted on the ring, considering uniform wear is,

$$\delta W = p_r.2\pi r.dr = \frac{C}{r} \times 2\pi r.dr = 2\pi C.dr$$

$\therefore$ Total load transmitted to the collar,

$$W = \int_{r_2}^{r_1} 2\pi C.dr = 2\pi C[r]_{r_2}^{r_1} = 2\pi C(r_1 - r_2)$$

We also know that frictional torque on the ring,

$$T_r = \mu.\delta W.r = \mu \times 2\pi C.dr.r = 2\pi\mu.C.r.dr$$

$\therefore$ Total frictional torque on the bearing,

$$T = \int_{r_2}^{r_1} 2\pi\mu C.r.dr = 2\pi\mu.C\left[\frac{r^2}{2}\right]_{r_2}^{r_1} = 2\pi\mu.C\left[\frac{(r_1)^2 - (r_2)^2}{2}\right]$$

$$= \pi\mu.C[(r_1)^2 - (r_2)^2]$$

Substituting the value of C from equation (*ii*),

$$T = \pi\mu \times \frac{W}{2\pi(r_1 - r_2)}[(r_1)^2 - (r_2)^2] = \frac{1}{2} \times \mu.W(r_1 + r_2)$$

10.31. Friction Clutches

A friction clutch has its principal application in the transmission of power of shafts and machines which must be started and stopped frequently.

In automobiles, friction clutch is used to connect the engine to the driven shaft. In operating such a clutch, care should be taken so that the friction surfaces engage easily and gradually brings the driven shaft up to proper speed.

The friction clutches of the following types are important from the subject point of view :

1. Disc or plate clutches (single disc or multiple disc clutch),

2. Cone clutches, and

3. Centrifugal clutches.

10.32. Single Disc or Plate Clutch

A single disc or plate clutch, as shown in Fig. 10.21, consists of a clutch plate whose both sides are faced with a friction material (usually of Ferrodo).

It is mounted on the hub which is free to move axially along the splines of the driven shaft. The pressure plate is mounted inside the clutch body which is bolted to the flywheel.

Both the pressure plate and the flywheel rotate with the enginecrankshaft or the driving shaft.

•

The pressure plate pushes the clutch plate towards the flywheel by a set of strong springs which are arranged radially inside the body.

The three levers (also known as release levers or fingers) are carried on pivots suspended from the case of the body. These are arranged in such a manner so that the pressure plate moves away from the flywheel by the inward movement of a thrust bearing.

The bearing is mounted upon a forked shaft and moves forward when the clutch pedal is pressed.

When the clutch pedal is pressed down, its linkage forces the thrust release bearing to move in towards the flywheel and pressing the longer ends of the levers inward. The levers are forced to turn on their suspended pivot and the pressure plate moves away from the flywheel by the knife edges, there by compressing the clutch springs. This action removes the pressure from the clutch plate and thus moves back from the flywheel and the driven shaft becomes stationary.

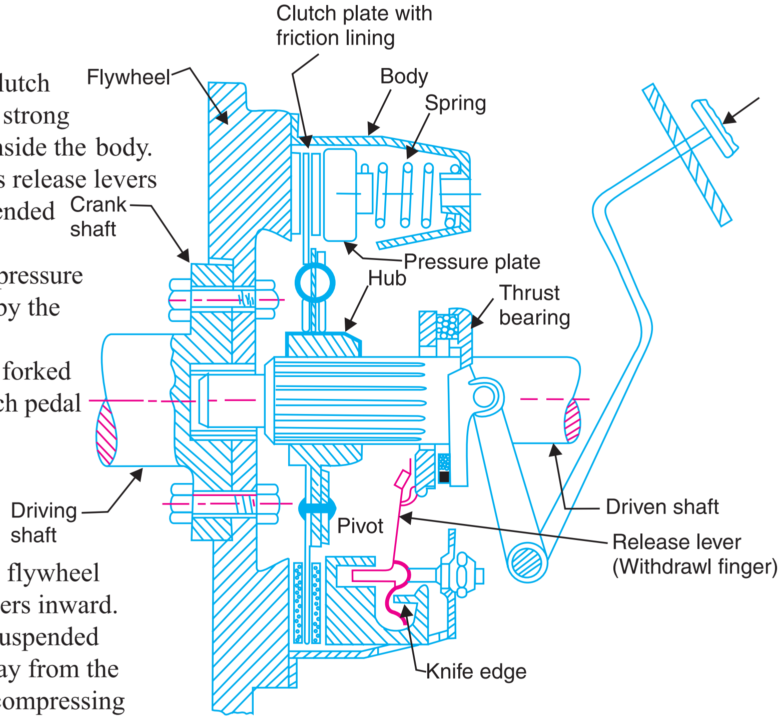


Fig. 10.21. Single disc or plate clutch.

On the other hand, when the foot is taken off from the clutch pedal, the thrust bearing moves back by the levers. This allows the springs to extend and thus the pressure plate pushes the clutch plate back towards the flywheel.

The axial pressure exerted by the spring provides a frictional force in the circumferential direction when the relative motion between the driving and driven members tends to take place. If the torque due to this frictional force exceeds the torque to be transmitted, then no slipping takes place and the power is transmitted from the driving shaft to the driven shaft.

Now consider two friction surfaces, maintained in contact by an axial thrust W, as shown in Fig. 10.22 (*a*).

Let T = Torque transmitted by the clutch,

p = Intensity of axial pressure with which the contact surfaces are held together,

r_1 and r_2 = External and internal radii of friction faces, and

μ = Coefficient of friction.

Consider an elementary ring of radius r and thickness dr as shown in Fig. 10.22 (*b*).

We know that area of contact surface or friction surface,

$$= 2\,\pi\, r.dr$$

$\therefore$ Normal or axial force on the ring,

$$\delta W = \text{Pressure} \times \text{Area} = p \times 2\,\pi\, r.dr$$

and the frictional force on the ring acting tangentially at radius r,

$$F_r = \mu.\delta W = \mu.p \times 2\,\pi\, r.dr$$

$\therefore$ Frictional torque acting on the ring,

$$T_r = F_r \times r = \mu.p \times 2\,\pi\, r.dr \times r = 2\,\pi \times \mu\, .p.r^2\, dr$$

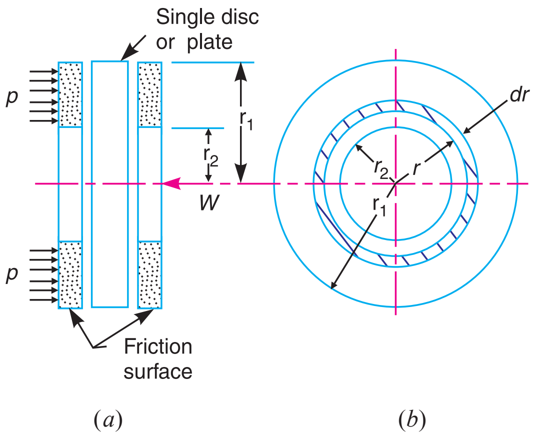


Fig. 10.22. Forces on a single disc or plate clutch.

We shall now consider the following two cases :

1. When there is a uniform pressure, and

2. When there is a uniform wear.

1. *Considering uniform pressure*

When the pressure is uniformly distributed over the entire area of the friction face, then the intensity of pressure,

$$p = \frac{W}{\pi[(r_1)^2 - (r_2)^2]} \qquad \ldots(i)$$

where W = Axial thrust with which the contact or friction surfaces are held together

We have discussed above that the frictional torque on the elementary ring of radius r and thickness dr is

$$T_r = 2\,\pi\,\mu.p.r^2\, dr$$

Integrating this equation within the limits from r_2 to r_1 for the total frictional torque.

$\therefore$ Total frictional torque acting on the friction surface or on the clutch,

$$T = \int_{r_1}^{r_2} 2\pi\mu.p.r^2.dr = 2\,\pi\mu\, p\left[\frac{r^3}{3}\right]_{r_2}^{r_1} = 2\pi\mu\, p\left[\frac{(r_1)^3 - (r_2)^3}{3}\right]$$

Substituting the value of p from equation (*i*),

$$T = 2\pi\mu \times \frac{W}{\pi[(r_1)^2 - (r_2)^2]} \times \frac{(r_1)^3 - (r_2)^3}{3}$$

$$= \frac{2}{3} \times \mu.W \left[\frac{(r_1)^3 - (r_2)^3}{(r_1)^2 - (r_2)^2}\right] = \mu.W.R$$

where R = Mean radius of friction surface

$$= \frac{2}{3}\left[\frac{(r_1)^3 - (r_2)^3}{(r_1)^2 - (r_2)^2}\right]$$

2. *Considering uniform wear*

In Fig. 10.22, let p be the normal intensity of pressure at a distance r from the axis of the clutch. Since the intensity of pressure varies inversely with the distance, therefore

$$p.r. = C \text{ (a constant)} \quad \text{or} \quad p = C/r \qquad ...(i)$$

and the normal force on the ring,

$$\delta W = p.2\pi r.dr = \frac{C}{r} \times 2\pi C.dr = 2\pi C.dr$$

∴ Total force acting on the friction surface,

$$W = \int_{r_2}^{r_1} 2\pi C\, dr = 2\pi C[r]_{r_2}^{r_1} = 2\pi C(r_1 - r_2)$$

or

$$C = \frac{W}{2\pi(r_1 - r_2)}$$

We know that the frictional torque acting on the ring,

$$T_r = 2\pi\mu.p\, r^2.dr = 2\pi\mu \times \frac{C}{r} \times r^2.dr = 2\pi\mu.C.r.dr \qquad ...(\because\ p = C/r)$$

∴ Total frictional torque on the friction surface,

$$T = \int_{r_2}^{r_1} 2\pi\mu.C.r.dr = 2\pi\mu.C\left[\frac{r^2}{2}\right]_{r_2}^{r_1} = 2\pi\mu.C\left[\frac{(r_1)^2 - (r_2)^2}{2}\right]$$

$$= \pi\mu.C[(r_1)^2 - (r_2)^2] = \pi\mu \times \frac{W}{2\pi(r_1 - r_2)}\left[(r_1)^2 - (r_2)^2\right]$$

$$= \frac{1}{2} \times \mu.W(r_1 + r_2) = \mu.W.R$$

where R = Mean radius of the friction surface $= \dfrac{r_1 + r_2}{2}$

10.33. Multiple Disc Clutch

A multiple disc clutch, as shown in Fig. 10.23, may be used when a lar ge torque is to be transmitted. The inside discs (usually of steel) are fastened to the driven shaft to permit axial motion (except for the last disc). The outside discs (usually of bronze) are held by bolts and are fastened to the housing which is keyed to the driving shaft. The multiple disc clutches are extensively used in motor cars, machine tools etc.

Let n_1 = Number of discs on the driving shaft, and

n_2 = Number of discs on the driven shaft.

∴ Number of pairs of contact surfaces,

$$n = n_1 + n_2 - 1$$

and total frictional torque acting on the friction surfaces or on the clutch,

$$T = n.\mu.W.R$$

where R = Mean radius of the friction surfaces

$$= \frac{2}{3}\left[\frac{(r_1)^3 - (r_2)^3}{(r_1)^2 - (r_2)^2}\right] \qquad \text{...(For uniform pressure)}$$

$$= \frac{r_1 + r_2}{2} \qquad \text{...(For uniform wear)}$$

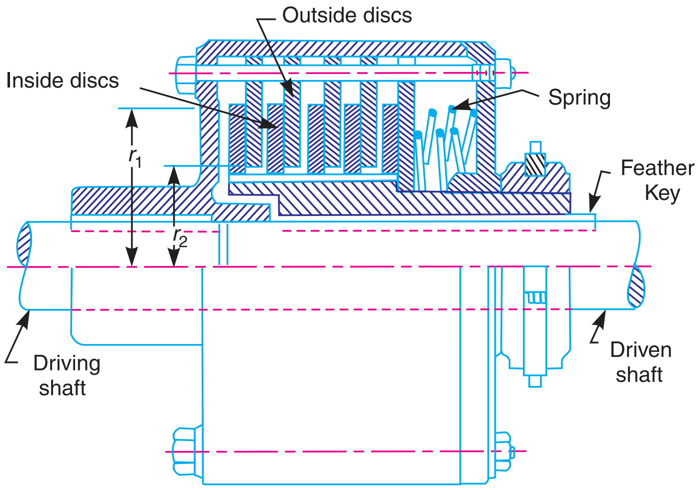


Fig. 10.23. Multiple disc clutch.

Example 4. *Determine the maximum, minimum and average pressure in plate clutch when the axial force is 4 kN. The inside radius of the contact surface is 50 mm and the outside radius is 100 mm. Assume uniform wear*

Solution. Given : $W = 4$ kN $= 4 \times 10^3$ N ; $r_2 = 50$ mm ; $r_1 = 100$ mm

Maximum pressure

Let $\quad p_{max}$ = Maximum pressure.

Since the intensity of pressure is maximum at the inner radius (r_2), therefore

$$p_{max} \times r_2 = C \quad \text{or} \quad C = 50\, p_{max}$$

We know that the total force on the contact surface (W),

$$4 \times 10^3 = 2\,\pi\, C\,(r_1 - r_2) = 2\,\pi \times 50\, p_{max}\,(100 - 50) = 15\,710\, p_{max}$$

$$\therefore \quad p_{max} = 4 \times 10^3/15\,710 = 0.2546 \text{ N/mm}^2 \quad \textbf{Ans.}$$

Minimum pressure

Let $\quad p_{min}$ = Minimum pressure.

Since the intensity of pressure is minimum at the outer radius (r_1), therefore

$$p_{min} \times r_1 = C \quad \text{or} \quad C = 100\, p_{min}$$

We know that the total force on the contact surface (W),

$$4 \times 10^3 = 2\,\pi\, C\,(r_1 - r_2) = 2\pi \times 100\, p_{min}\,(100 - 50) = 31\,420\, p_{min}$$

$$\therefore \quad p_{min} = 4 \times 10^3/31\,420 = 0.1273 \text{ N/mm}^2 \quad \textbf{Ans.}$$

Average pressure

We know that average pressure,

$$p_{av} = \frac{\text{Total normal force on contact surface}}{\text{Cross-sectional area of contact surfaces}}$$

$$= \frac{W}{\pi[(r_1)^2 - (r_2)^2]} = \frac{4 \times 10^3}{\pi[(100)^2 - (50)^2]} = 0.17 \text{ N/mm}^2 \quad \textbf{Ans.}$$

Example 5. *A single plate clutch, with both sides effective, has outer and inner diameters 300 mm and 200 mm respectively. The maximum intensity of pressure at any point in the contact surface is not to exceed 0.1 N/mm². If the coefficient of friction is 0.3, determine the power transmitted by a clutch at a speed 2500 r.p.m.*

Solution. Given : $d_1 = 300$ mm or $r_1 = 150$ mm ; $d_2 = 200$ mm or $r_2 = 100$ mm ; $p = 0.1$ N/mm² ; $\mu = 0.3$; $N = 2500$ r.p.m. or $\omega = 2\pi \times 2500/60 = 261.8$ rad/s

Since the intensity of pressure (p) is maximum at the inner radius (r_2), therefore for uniform wear,

$$p.r_2 = C \quad \text{or} \quad C = 0.1 \times 100 = 10 \text{ N/mm}$$

We know that the axial thrust,

$$W = 2\,\pi\, C\,(r_1 - r_2) = 2\,\pi \times 10\,(150 - 100) = 3142 \text{ N}$$

and mean radius of the friction surfaces for uniform wear,

$$R = \frac{r_1 + r_2}{2} = \frac{150 + 100}{2} = 125 \text{ mm} = 0.125 \text{ m}$$

We know that torque transmitted,

$$T = n.\mu.W.R = 2 \times 0.3 \times 3142 \times 0.125 = 235.65 \text{ N-m}$$

...($\because$ $n = 2$, for both sides of plate effective)

$\therefore$ Power transmitted by a clutch,

$$P = T.\omega = 235.65 \times 261.8 = 61\ 693 \text{ W} = 61.693 \text{ kW}$$ **Ans.**

10.34. Cone Clutch

A cone clutch, as shown in Fig. 10.24, was extensively used in automobiles but now-a-days it has been replaced completely by the disc clutch.

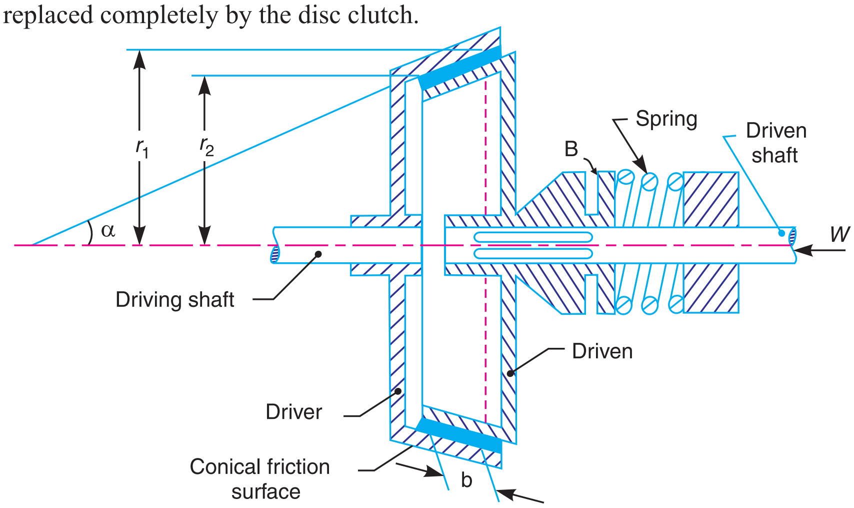


Fig. 10.24. Cone clutch.

It consists of one pair of friction surface only . In a cone clutch, the driver is keyed to the driving shaft by a sunk key and has an inside conical surface or face which exactly fits into the outside conical surface of the driven. The driven member resting on the feather key in the driven shaft, may be shifted along the shaft by a forked lever provided at *B*, in order to engage the clutch by bringing the two conical surfaces in contact. Due to the frictional resistance set up at this contact surface, the torque is transmitted from one shaft to another. In some cases, a spring is placed around the driven shaft in contact with the hub of the driven. This spring holds the clutch faces in contact and maintains the pressure between them, and the forked lever is used only for disengagement of the clutch. The contact surfaces of the clutch may be metal to metal contact, but more often the driven member is lined with some material like wood, leather, cork or asbestos etc. The material of the clutch faces (*i.e.* contact surfaces) depends upon the allowable normal pressure and the coefficient of friction.

Consider a pair of friction surface as shown in Fig. 10.25 (*a*). Since the area of contact of a pair of friction surface is a frustrum of a cone, therefore the torque transmitted by the cone clutch may be determined in the similar manner as discussed for conical pivot bearings in Art. 10.28.

Let p_n = Intensity of pressure with which the conical friction surfaces are held together (*i.e.* normal pressure between contact surfaces),

r_1 and r_2 = Outer and inner radius of friction surfaces respectively.

R = Mean radius of the friction surface $= \dfrac{r_1 + r_2}{2}$,

α = Semi angle of the cone (also called face angle of the cone) or the angle of the friction surface with the axis of the clutch,

μ = Coefficient of friction between contact surfaces, and

b = Width of the contact surfaces (also known as face width or clutch face).

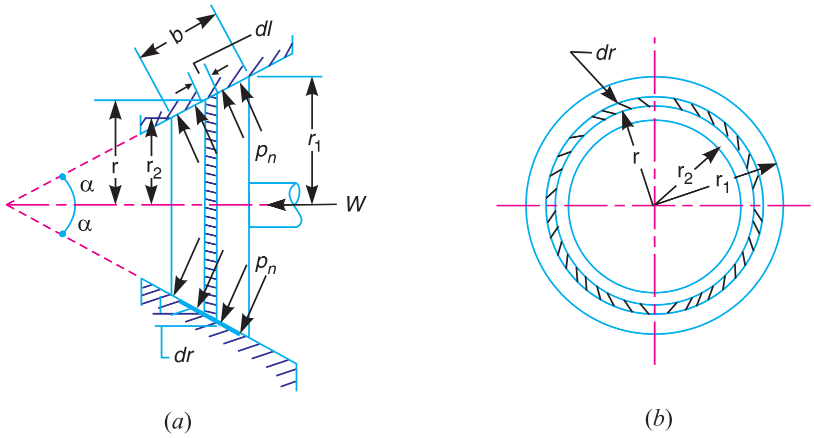


Fig. 10.25. Friction surfaces as a frustrum of a cone.

Consider a small ring of radius r and thickness dr, as shown in Fig. 10.25 (*b*). Let dl is length of ring of the friction surface, such that

$$dl = dr.\text{cosec}\ \alpha$$

∴ Area of the ring,

$$A = 2\pi\, r.dl = 2\pi r.dr\ \text{cosec}\ \alpha$$

1. *Considering uniform pressure*

We know that normal load acting on the ring,

$$\delta W_n = \text{Normal pressure} \times \text{Area of ring} = p_n \times 2\,\pi\, r.dr.\text{cosec}\ \alpha$$

and the axial load acting on the ring,

$$\delta W = \text{Horizontal component of } \delta W_n \ (i.e. \text{ in the direction of } W)$$

$$= \delta W_n \times \sin\alpha = p_n \times 2\pi\, r.dr.\ \text{cosec}\ \alpha \times \sin\alpha = 2\pi \times p_n.r.dr$$

∴ Total axial load transmitted to the clutch or the axial spring force required,

$$W = \int_{r_2}^{r_1} 2\pi\, p_n.r.dr = 2\pi\, p_n \left[\frac{r^2}{2}\right]_{r_2}^{r_1} = 2\pi\, p_n \left[\frac{(r_1)^2 - (r_2)^2}{2}\right]$$

$$= \pi\, p_n \left[(r_1)^2 - (r_2)^2\right]$$

$$\therefore \qquad p_n = \frac{W}{\pi[(r_1)^2 - (r_2)^2]} \qquad \ldots(i)$$

We know that frictional force on the ring acting tangentially at radius r,

$$F_r = \mu.\delta W_n = \mu.p_n \times 2\,\pi\, r.dr.\text{cosec}\ \alpha$$

∴ Frictional torque acting on the ring,

$$T_r = F_r \times r = \mu.p_n \times 2\,\pi\, r.dr.\ \text{cosec}\ \alpha.r = 2\,\pi\, \mu.p_n.\text{cosec}\ \alpha.r^2\, dr$$

Integrating this expression within the limits from r_2 to r_1 for the total frictional torque on the clutch.

∴ Total frictional torque,

$$T = \int_{r_2}^{r_1} 2\,\pi\mu.p_n.\text{cosec}\ \alpha.r^2.dr = 2\,\pi\mu\, p_n.\text{cosec}\ \alpha \left[\frac{r^3}{3}\right]_{r_2}^{r_1}$$

$$= 2\pi\,\mu\, p_n.\text{cosec}\ \alpha \left[\frac{(r_1)^3 - (r_2)^3}{3}\right]$$

Substituting the value of p_n from equation (*i*), we get

$$T = 2\pi\,\mu \times \frac{W}{\pi\,[(r_1)^2 - (r_2)^2]} \times \text{cosec}\ \alpha \left[\frac{(r_1)^3 - (r_2)^3}{3}\right]$$

$$= \frac{2}{3} \times \mu.W.\text{cosec}\ \alpha \left[\frac{(r_1)^3 - (r_2)^3}{(r_1)^2 - (r_2)^2}\right] \qquad ..(ii)$$

2. *Considering uniform wear*

In Fig. 10.25, let p_r be the normal intensity of pressure at a distance r from the axis of the clutch. We know that, in case of uniform wear , the intensity of pressure varies inversely with the distance.

$$\therefore \qquad p_r.r = C \text{ (a constant)} \quad \text{or} \quad p_r = C/r$$

We know that the normal load acting on the ring,

$$\delta W_n = \text{Normal pressure} \times \text{Area of ring} = p_r \times 2\pi r.dr \operatorname{cosec} \alpha$$

and the axial load acting on the ring ,

$$\delta W = \delta W_n \times \sin\alpha = p_r.2\,\pi\, r.dr.\operatorname{cosec}\alpha\,.\sin\alpha = p_r \times 2\,\pi\, r.dr$$

$$= \frac{C}{r} \times 2\pi r.dr = 2\pi C.dr \qquad \ldots(\because\ p_r = C/r)$$

$\therefore$ Total axial load transmitted to the clutch,

$$W = \int_{r_2}^{r_1} 2\pi C.dr = 2\pi C[r]_{r_2}^{r_1} = 2\pi C(r_1 - r_2)$$

or
$$C = \frac{W}{2\pi(r_1 - r_2)} \qquad \ldots(iii)$$

We know that frictional force acting on the ring,

$$F_r = \mu.\delta W_n = \mu.p_r \times 2\,\pi\, r \times dr \operatorname{cosec}\alpha$$

and frictional torque acting on the ring,

$$T_r = F_r \times r = \mu.p_r \times 2\,\pi\, r.dr.\operatorname{cosec}\alpha \times r$$

$$= \mu \times \frac{C}{r} \times 2\pi r^2.dr.\operatorname{cosec}\alpha = 2\pi\mu.C \operatorname{cosec}\alpha \times r\ dr$$

$\therefore$ Total frictional torque acting on the clutch,

$$T = \int_{r_2}^{r_1} 2\pi\mu.C.\operatorname{cosec}\alpha.r\,dr = 2\pi\mu.C.\operatorname{cosec}\alpha\left[\frac{r^2}{2}\right]_{r_2}^{r_1}$$

$$= 2\pi\mu.C.\operatorname{cosec}\alpha\left[\frac{(r_1)^2 - (r_2)^2}{2}\right]$$

Substituting the value of C from equation ***(i)***, we have

$$T = 2\pi\mu \times \frac{W}{2\pi(r_1 - r_2)} \times \operatorname{cosec}\alpha\left[\frac{(r_1)^2 - (r_2)^2}{2}\right]$$

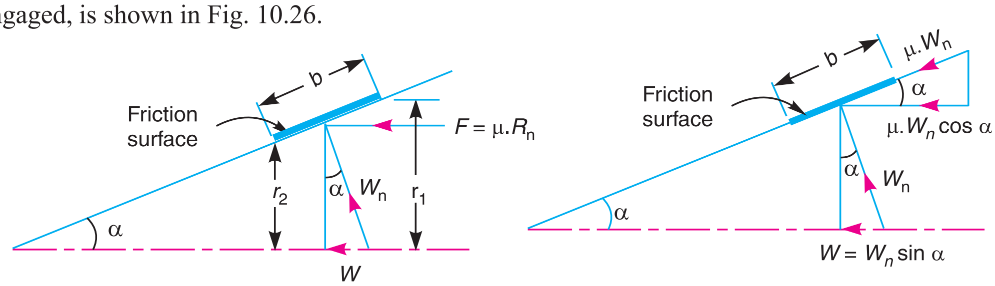
$$= \mu.W \operatorname{cosec}\alpha\left(\frac{r_1 + r_2}{2}\right) = \mu.W.R \operatorname{cosec}\alpha \qquad \ldots(iv)$$

where
$$R = \frac{r_1 + r_2}{2} = \text{Mean radius of friction surface}$$

Since the normal force acting on the friction surface, $W_n = W/\sin\alpha$, therefore the equation ***(iv)*** may be written as

$$T = \mu.W_n.R \qquad \ldots(v)$$

The forces on a friction surface, for steady operation of the clutch and after the clutch is engaged, is shown in Fig. 10.26.



(*a*) For steady operation of the clutch

(*b*) During engagement of the clutch.

Fig. 10.26. Forces on a friction surface.

$$r_1 - r_2 = b \sin \alpha; \text{ and } R = \frac{r_1 + r_2}{2} \quad \text{or} \quad r_1 + r_2 = 2R$$

∴ From equation, ***(i)***, normal pressure acting on the friction surface,

$$p_n = \frac{W}{\pi[(r_1)^2 - (r_2)^2]} = \frac{W}{\pi(r_1 + r_2)(r_1 - r_2)} = \frac{W}{2\pi R.b.\sin\alpha}$$

or $\quad W = p_n \times 2\pi R.b \sin\alpha = W_n \sin\alpha$

where $\quad W_n$ = Normal load acting on the friction surface $= p_n \times 2\pi R.b$

Now the equation ***(iv)*** may be written as,

$$T = \mu(p_n \times 2\pi R.b \sin\alpha)\, R \operatorname{cosec} \alpha = 2\pi\mu.p_n.R^2 b$$

Example 6. *A conical friction clutch is used to transmit 90 kW at 1500 r.p.m. The semi-cone angle is 20° and the coefficient of friction is 0.2. If the mean diameter of the bearing surface is 375 mm and the intensity of normal pressure is not to exceed 0.25 N/mm², find the dimensions of the conical bearing surface and the axial load required.*

Solution. Given : $P = 90$ kW $= 90 \times 10^3$ W ; $N = 1500$ r.p.m. or $\omega = 2\pi \times 1500/60 = 156$ rad/s ; $\alpha = 20°$; $\mu = 0.2$; $D = 375$ mm or $R = 187.5$ mm ; $p_n = 0.25$ N/mm²

Dimensions of the conical bearing surface

Let $\quad r_1$ and r_2 = External and internal radii of the bearing surface respectively,

b = Width of the bearing surface in mm, and

T = Torque transmitted.

We know that power transmitted (P),

$$90 \times 10^3 = T.\omega = T \times 156$$

$$\therefore \quad T = 90 \times 10^3/156 = 577 \text{ N-m} = 577 \times 10^3 \text{ N-mm}$$

and the torque transmitted (T),

$$577 \times 10^3 = 2\pi\mu\, p_n.R^2.b = 2\pi \times 0.2 \times 0.25\,(187.5)^2\, b = 11\,046\, b$$

$$\therefore \quad b = 577 \times 10^3/11\,046 = 52.2 \text{ mm}$$ **Ans.**

We know that $r_1 + r_2 = 2R = 2 \times 187.5 = 375$ mm ...***(i)***

and $\quad r_1 - r_2 = b \sin\alpha = 52.2 \sin 20° = 18$ mm ...***(ii)***

From equations ***(i)*** and ***(ii)***,

$r_1 = 196.5$ mm, and $r_2 = 178.5$ mm **Ans.**

Axial load required

Since in case of friction clutch, uniform wear is considered and the intensity of pressure is maximum at the minimum contact surface radius (r_2), therefore

$$p_n.r_2 = C \text{ (a constant) or } C = 0.25 \times 178.5 = 44.6 \text{ N/mm}$$

We know that the axial load required, $W = 2\pi C(r_1 - r_2) = 2\pi \times 44.6\,(196.5 - 178.5) = 5045$ N **Ans**

10.35. Centrifugal Clutch

The centrifugal clutches are usually incorporated into the motor pulleys. It consists of a number of shoes on the inside of a rim of the pulley as shown in Fig. 10.28.

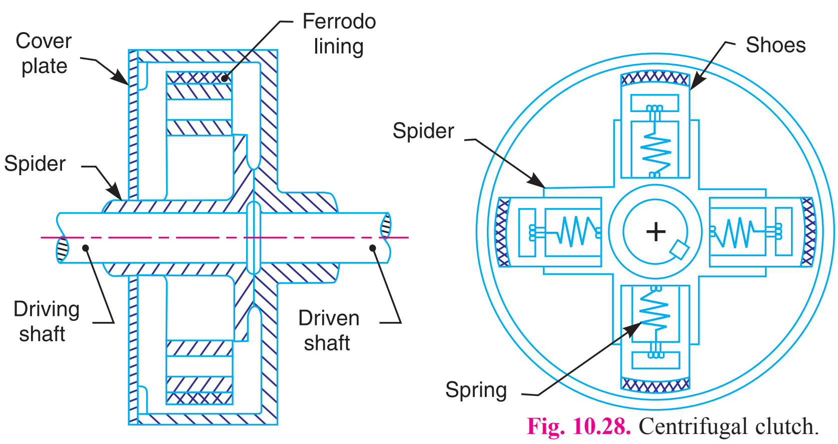


Fig. 10.28. Centrifugal clutch.

The outer surface of the shoes are covered with a friction material. These shoes, which can move radially in guides, are held against the boss (or spider) on the driving shaft by means of springs. The springs exert a radially inward force which is assumed constant. The mass of the shoe, when revolving, causes it to exert a radially outward force (*i.e.* centrifugal force). The magnitude of this centrifugal force depends upon the speed at which the shoe is revolving. A little consideration will show that when the centrifugal force is less than the spring force, the shoe remains in the same position as when the driving shaft was stationary, but when the centrifugal force is equal to the spring force, the shoe is just floating. When the centrifugal force exceeds the spring force, the shoe moves outward and comes into contact with the driven member and presses against it. The force with which the shoe presses against the driven member is the difference of the centrifugal force and the spring force. The increase of speed causes the shoe to press harder and enables more torque to be transmitted.

In order to determine the mass and size of the shoes, the following procedure is adopted :

1. *Mass of the shoes*

Consider one shoe of a centrifugal clutch as shown in Fig. 10.29.

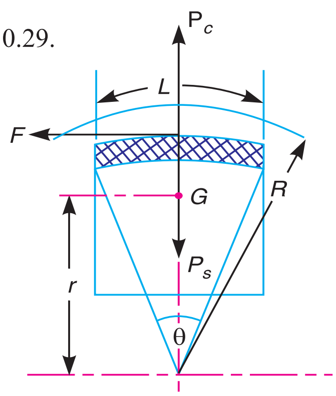


Fig. 10.29. Forces on a shoe of centrifugal clutch.

Let m = Mass of each shoe,

n = Number of shoes,

r = Distance of centre of gravity of the shoe from the centre of the spider,

R = Inside radius of the pulley rim,

N = Running speed of the pulley in r.p.m.,

ω = Angular running speed of the pulley in rad/s = $2\pi N/60$ rad/s,

ω_1 = Angular speed at which the engagement begins to take place, and

μ = Coefficient of friction between the shoe and rim.

We know that the centrifugal force acting on each shoe at the running speed,

$$*P_c = m.\omega^2.r$$

and the inward force on each shoe exerted by the spring at the speed at which engagement begins to take place,

$$P_s = m\,(\omega_1)^2\,r$$

∴ The net outward radial force (*i.e.* centrifugal force) with which the shoe presses against the rim at the running speed

$$= P_c - P_s$$

and the frictional force acting tangentially on each shoe,

$$F = \mu\,(P_c - P_s)$$

∴ Frictional torque acting on each shoe,

$$= F \times R = \mu\,(P_c - P_s)\,R$$

and total frictional torque transmitted,

$$T = \mu\,(P_c - P_s)\,R \times n = n.F.R$$

From this expression, the mass of the shoes (m) may be evaluated.

2. *Size of the shoes*

Let l = Contact length of the shoes,

b = Width of the shoes,

R = Contact radius of the shoes. It is same as the inside radius of the rim of the pulley.

θ = Angle subtended by the shoes at the centre of the spider in radians.

p = Intensity of pressure exerted on the shoe. In order to ensure reasonable life, the intensity of pressure may be taken as 0.1 N/mm².

We know that $\theta = l/R$ rad or $l = \theta.R$

$\therefore$ Area of contact of the shoe,

$$A = l.b$$

and the force with which the shoe presses against the rim

$$= A \times p = l.b.p$$

Since the force with which the shoe presses against the rim at the running speed is $(P_c - P_s)$, therefore

$$l.b.p = P_c - P_s$$

From this expression, the width of shoe (b) may be obtained.

Example 7. *A centrifugal clutch is to transmit 15 kW at 900 r.p.m. The shoes are four in number. The speed at which the engagement begins is 3/4th of the running speed. The inside radius of the pulley rim is 150 mm and the centre of gravity of the shoe lies at 120 mm from the centre of the spider. The shoes are lined with Ferrodo for which the coefficient of friction may be taken as 0.25. Determine : **1.** Mass of the shoes, and **2.** Size of the shoes, if angle subtended by the shoes at the centre of the spider is 60° and the pressure exerted on the shoes is 0.1 N/mm².*

Solution. Given : $P = 15$ kW $= 15 \times 10^3$ W ; $N = 900$ r.p.m. or $\omega = 25 \times 900/60 = 94.26$ rad/s ; $n = 4$; $R = 150$ mm $= 0.15$ m ; $r = 120$ mm $= 0.12$ m ; $\mu = 0.25$

Since the speed at which the engagement begins (*i.e.* ω_1) is 3/4th of the running speed (*i.e.* ω), therefore

$$\omega_1 = \frac{3}{4}\,\omega = \frac{3}{4} \times 94.26 = 70.7 \text{ rad/s}$$

Let T = Torque transmitted at the running speed.

We know that power transmitted (P),

$$15 \times 10^3 = T.\omega = T \times 94.26 \quad \text{or} \quad T = 15 \times 10^3/94.26 = 159 \text{ N-m}$$

1. *Mass of the shoes*

Let m = Mass of the shoes in kg.

We know that the centrifugal force acting on each shoe,

$$P_c = m.\omega^2.r = m\,(94.26)^2 \times 0.12 = 1066\,m \text{ N}$$

and the inward force on each shoe exerted by the spring *i.e.* the centrifugal force at the engagement speed ω_1,

$$P_s = m\,(\omega_1)^2\,r = m\,(70.7)^2 \times 0.12 = 600\,m \text{ N}$$

$\therefore$ Frictional force acting tangentially on each shoe,

$$F = \mu\,(P_c - P_s) = 0.25\,(1066\,m - 600\,m) = 116.5\,m \text{ N}$$

We know that the torque transmitted (T),

$$159 = n.F.R = 4 \times 116.5\,m \times 0.15 = 70\,m \quad \text{or} \quad m = 2.27 \text{ kg}$$ **Ans.**

2. *Size of the shoes*

Let l = Contact length of shoes in mm,

b = Width of the shoes in mm,

θ = Angle subtended by the shoes at the centre of the spider in radians
$= 60° = \pi/3$ rad, and ...(Given)

p = Pressure exerted on the shoes in N/mm² = 0.1 N/mm² ...(Given)

We know that $l = \theta.\,R = \dfrac{\pi}{3} \times 150 = 157.1$ mm

and $l.b.p = P_c - P_s = 1066\,m - 600\,m = 466\,m$

$\therefore$ $157.1 \times b \times 0.1 = 466 \times 2.27 = 1058$

or $b = 1058/157.1 \times 0.1 = 67.3$ mm **Ans.**

EXERCISES

UNIT-III BRAKES AND DYNAMOMETERS

19.1. Introduction

A ***brake*** is a device by means of which artificial frictional resistance is applied to a moving machine member, in order to retard or stop the motion of a machine. The energy absorbed by brakes is dissipated in the form of heat.

The capacity of a brake depends upon the following factors :

1. The unit pressure between the braking surfaces,
2. The coefficient of friction between the braking surfaces,
3. The peripheral velocity of the brake drum,
4. The projected area of the friction surfaces, and
5. The ability of the brake to dissipate heat equivalent to the energy being absorbed.

19.2. Materials for Brake Lining

The material used for the brake lining should have the following characteristics :

1.It should have high coefficient of friction with minimum fading. In other words, the coeffi-cient of friction should remain constant with change in temperature.

2.It should have low wear rate.

3.It should have high heat resistance.

4.It should have high heat dissipation capacity.

5.It should have adequate mechanical strength.

6.It should not be affected by moisture and oil.

Types of Brakes:

The brakes, according to the means used for transforming the energy by the braking elements,are classified as :

1. Hydraulic brakes e.g. pumps or hydrodynamic brakeand fluid agitator,
2. Electric brakes e.g. generators and eddy currentbrakes,
3. Mechanical brakes.

The mechanical brakes, according to the direction ofacting force, may be divided into the following two groups :

(a) **Radial brakes:** In these brakes, the force acting onthe brake drum is in radial direction.

The radial brakes may be sub-divided into external brakes and internal brakes.

According to the shape of the friction elements, these brakes may be block or shoe brakes and band brakes.

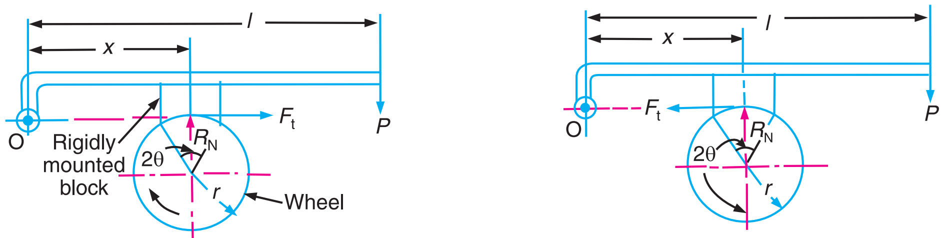
(b) **Axial brakes:** In these brakes, the force acting on the brake drum is in axial direction.

The axial brakes may be disc brakes and cone brakes.

19.4. Single Block or Shoe Brake

A single block or shoe brake is consists of a block or shoe which is pressed against the rim of a revolving brake wheel drum. The block is made of a softer material thanthe rim of the wheel. This type of a brake is commonly used on railway trains and tram cars.

The friction between the block and the wheel causes a tangential braking force to act on the wheel, which retard the rotation of the wheel. The block is pressed against the wheel by a force applied to one endof a lever to which the block is rigidly fixed as shown in Fig. 19.1. The other end of the lever is pivoted on a fixed fulcrum *O*.



(*a*) Clockwise rotation of brake wheel (*b*) Anticlockwise rotation of brake wheel.

Fig. 19.1. Single block brake. Line of action of tangential force passes through the fulcrum of the lever

Let P = Force applied at the end of the lever,

R_N = Normal force pressing the brake block on the wheel,

r = Radius of the wheel,

2θ = Angle of contact surface of the block,

μ = Coefficient of friction, and

F_t = Tangential braking force or the frictional force acting at the contact surface of the block and the wheel.

If the angle of contact is less than 60°, then it may be assumed that the normal pressure between the block and the wheel is uniform. In such cases, tangential braking force on the wheel,

$$F_t = \mu.R_N \quad \text{...}(i)$$

and the braking torque, $T_B = F_t.r = \mu.R_N.r$ ... (*ii*)

Let us now consider the following three cases :

Case 1. When the line of action of tangential braking force (F_t) passes through the fulcrum O of the lever, and the brake wheel rotates clockwise as shown in Fig. 19.1(*a*), then for equilibrium, taking moments about the fulcrum O, we have

$$R_N \times x = P \times l \quad \text{or} \quad R_N = \frac{P \times l}{x}$$

∴ Braking torque,

$$T_B = \mu.R_N.r = \mu \times \frac{P.l}{x} \times r = \frac{\mu.P.l.r}{x}$$

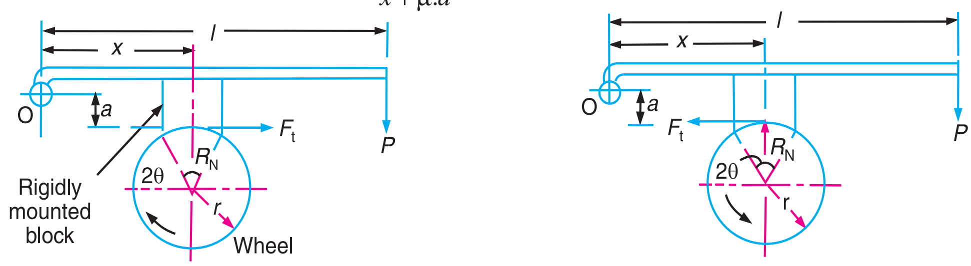
It may be noted that when the brake wheel rotates anticlockwise as shown in Fig. 19.1 (*b*), then the braking torque is same, *i.e.*

$$T_B = \mu.R_N.r = \frac{\mu.P.l.r}{x}$$

Case 2. When the line of action of the tangential braking force (F_t) passes through a distance '*a*' below the fulcrum O, and the brake wheel rotates clockwise as shown in Fig. 19.2 (*a*), then for equilibrium, taking moments about the fulcrum O,

$$R_N \times x + F_t \times a = P.l \quad \text{or} \quad R_N \times x + \mu R_N \times a = P.l \quad \text{or} \quad R_N = \frac{P.l}{x + \mu.a}$$

and braking torque, $T_B = \mu R_N.r = \dfrac{\mu.p.l.r}{x + \mu.a}$



(*a*) Clockwise rotation of brake wheel. (*b*) Anticlockwise rotation of brake wheel.

Fig. 19.2. Single block brake. Line of action of F_t passes below the fulcrum.

When the brake wheel rotates anticlockwise, as shown in Fig. 19.2 (*b*), then for equilibrium,

$$R_N.x = P.l + F_t.a = P.l + \mu.R_N.a \quad \text{...}(i)$$

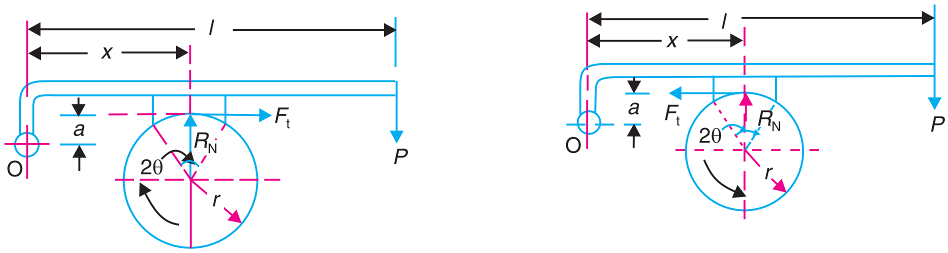
or $R_N (x - \mu.a) = P.l$ or $R_N = \dfrac{P.l}{x - \mu.a}$

and braking torque, $T_B = \mu.R_N.r = \dfrac{\mu.P.l.r}{x - \mu.a}$

Case 3. When the line of action of the tangential braking force (F_t) passes through a distance '*a*' above the fulcrum O, and the brake wheel rotates clockwise as shown in Fig. 19.3 (*a*), then for equilibrium, taking moments about the fulcrum O, we have

$$R_N.x = P.l + F_t \,.\, a = P.l + \mu.R_N.a \quad \text{. . . }(ii)$$

or $R_N (x - \mu.a) = P.l$ or $R_N = \dfrac{P.l}{x - \mu.a}$



(*a*) Clockwise rotation of brake wheel. (*b*) Anticlockwise rotation of brake wheel.

Fig. 19.3. Single block brake. Line of action of F_t passes above the fulcrum.

and braking torque, $T_B = \mu . R_N . r = \dfrac{\mu . P . l . r}{x - \mu . a}$

When the brake wheel rotates anticlockwise as shown in Fig. 19.3 (*b*), then for equilibrium, taking moments about the fulcrum O, we have

$$R_N \times x + F_t \times a = P.l \quad \text{or} \quad R_N \times x + \mu . R_N \times a = P.l \quad \text{or} \quad R_N = \frac{P.l}{x + \mu . a}$$

and braking torque, $T_B = \mu . R_N . r = \dfrac{\mu . P . l . r}{x + \mu . a}$

19.5. Pivoted Block or Shoe Brake

We have discussed in the previous article that when the angle of contact is less than 60°, then it may be assumed that the normal pressure between the block and the wheel is uniform. But when the angle of contact is greater than 60°, then the unit pressure normal to the surface of contact is less at the ends than at the centre. This gives uniform wear of the brake lining in the direction of the applied force. The braking torque for a pivoted block or shoe brake (*i.e.* when $2\theta > 60°$) is given by

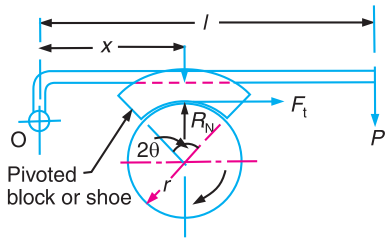


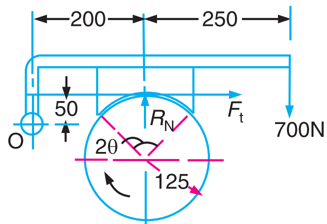
Fig. 19.4. Pivoted block or shoe brake.

$$T_B = F_t \times r = \mu' . R_N . r$$

where μ' = Equivalent coefficient of friction = $\dfrac{4\mu \sin\theta}{2\theta + \sin 2\theta}$, and

μ = Actual coefficient of friction.

Example 1. *A single block brake is shown in Fig. 19.5. The diameter of the drum is 250 mm and the angle of contact is 90°. If the operating force of 700 N is applied at the end of a lever and the coefficient of friction between the drum and the lining is 0.35, determine the torque that may be transmitted by the block brake.*



All dimensions in mm.

Fig. 19.5

Solution. Given : d = 250 mm or r = 125 mm ; $2\theta = 90°$ $= \pi/2$ rad ; P = 700 N ; μ = 0.35

Since the angle of contact is greater than 60°, therefore equivalent coefficient of friction,

$$\mu' = \frac{4\mu \sin\theta}{2\theta + \sin 2\theta} = \frac{4 \times 0.35 \times \sin 45°}{\pi/2 + \sin 90°} = 0.385$$

Let R_N = Normal force pressing the block to the brake drum, and

F_t = Tangential braking force = $\mu' . R_N$

Taking moments about the fulcrum O, we have

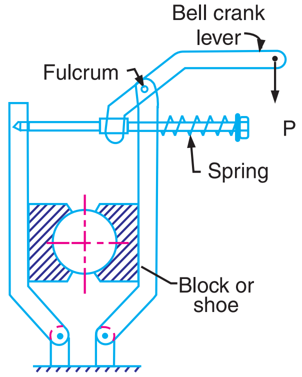
$$700(250 + 200) + F_t \times 50 = R_N \times 200 = \frac{F_t}{\mu'} \times 200 = \frac{F_t}{0.385} \times 200 = 520\ F_t$$

or $520\ F_t - 50 F_t = 700 \times 450$ or $F_t = 700 \times 450/470 = 670$ N

We know that torque transmitted by the block brake,

$T_B = F_t \times r = 670 \times 125 = 8\ 3750$ N-mm = 83.75N-m **Ans.**

19.6. Double Block or Shoe Brake



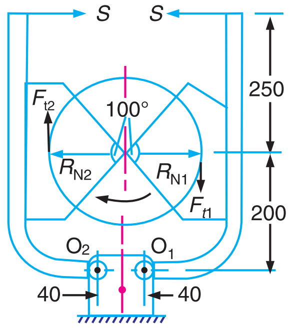
It consists of two brake blocks applied at the opposite ends of a diameter of the wheel which eliminate or reduces the unbalanced force on the shaft. The brake is set by a spring which pulls the upper ends of the brake arms together. When a force P is applied to the bellcrank lever, the spring is compressed and the brake is released.

In a double block brake, the braking action is doubled by the use of two blocks and these blocks may be operated practically by the same force which will operate one. In case of double block or shoe brake, the braking torque is given by

$$T_B = (F_{t1} + F_{t2})\, r$$

where F_{t1} and F_{t2} are the braking forces on the two blocks.

Example 2. *A double shoe brake, as shown in Fig. 19.10, is capable of absorbing a torque of 1400 N-m. The diameter of the brake drum is 350 mm and the angle of contact for each shoe is 100°. If the coefficient of friction between the brake drum and lining is 0.4 ; find* ***1.*** *the spring force necessary to set the brake ; and* ***2.*** *the width of the brake shoes, if the bearing pressure on the lining material is not to exceed 0.3 N/mm².*



Solution. Given : $T_B = 1400$ N-m $= 1400 \times 10^3$ N-mm ; $d = 350$ mm or $r = 175$ mm ; $2\theta = 100° = 100 \times \pi/180 = 1.75$ rad; $\mu = 0.4$; $p_b = 0.3$ N/mm²

1. *Spring force necessary to set the brake*

Let S = Spring force necessary to set the brake

R_{N1} and F_{t1} = Normal reaction and the braking force on the right hand side shoe, and

R_{N2} and F_{t2} = Corresponding values on the left hand side shoe.

Since the angle of contact is greater than 60°, therefore equivalent coefficient of friction,

$$\mu' = \frac{4\mu \sin\theta}{2\theta + \sin 2\theta} = \frac{4 \times 0.4 \times \sin 50°}{1.75 + \sin 100°} = 0.45$$

Taking moments about the fulcrum O_1, we have

$$S \times 450 = R_{N1} \times 200 + F_{t1}(175 - 40) = \frac{F_{t1}}{0.45} \times 200 + F_{t1} \times 135 = 579.4\, F_{t1} \quad \left(\text{Substituting } R_{N1} = \frac{F_{t1}}{\mu'}\right)$$

$\therefore \quad F_{t1} = S \times 450 / 579.4 = 0.776\, S \quad \ldots$

Again taking moments about O_2, we have

$$S \times 450 + F_{t2}(175 - 40) = R_{N2} \times 200 = \frac{F_{t2}}{0.45} \times 200 = 444.4\, F_{t2} \quad \left(\text{Substituting } R_{N2} = \frac{F_{t2}}{\mu'}\right)$$

$444.4\, F_{t2} - 135 F_{t2} = S \times 450$ or $309.4\, F_{t2} = S \times 450$

$\therefore \quad F_{t2} = S \times 450 / 309.4 = 1.454\, S \quad \ldots$

We know that torque capacity of the brake (T_B),

$$1400 \times 10^3 = (F_{t1} + F_{t2})\, r = (0.776\, S + 1.454\, S)\, 175 = 390.25\, S$$

$\therefore \quad S = 1400 \times 10^3/390.25 = 3587$ N **Ans.**

2. *Width of the brake shoes*

Let b = Width of the brake shoes in mm.

We know that projected bearing area for one shoe,

$$A_b = b(2r \sin\theta) = b(2 \times 175 \sin 50°) = 268\, b \text{ mm}^2$$

Normal force on the right hand side of the shoe,

$$R_{N1} = \frac{F_{t1}}{\mu'} = \frac{0.776 \times S}{0.45} = \frac{0.776 \times 3587}{0.45} = 6186 \text{ N}$$

and normal force on the left hand side of the shoe,

$$R_{N2} = \frac{F_{t2}}{\mu'} = \frac{1.454 \times S}{0.45} = \frac{1.454 \times 3587}{0.45} = 11\,590 \text{ N}$$

We see that the maximum normal force is on the left hand side of the shoe. Therefore we shall find the width of the shoe for the maximum normal force *i.e.* R_{N2}.

We know that the bearing pressure on the lining material (p_b),

$$0.3 = \frac{R_{N2}}{A_b} = \frac{11\ 590}{268\ b} = \frac{43.25}{b}$$

$\therefore$ $b = 43.25 / 0.3 = 144.2$ mm **Ans.**

19.7. Simple Band Brake

A band brake consists of a flexible band of leather, one or more ropes, or a steel lined with friction material, which embraces a part of the circumference of the drum. A band brake, as shown in Fig. 19.11, is called a ***simple band brake*** in which one end of the band is attached to a fixed pin or fulcrum of the lever while the other end is attached to the lever at a distance b from the fulcrum.

When a force P is applied to the lever at C, the lever turns about the fulcrum pin O and tightens the band on the drum and hence the brakes are applied. The friction between the band and the drum provides the braking force. The force P on the lever at C may be determined as discussed below :

Let T_1 = Tension in the tight side of the band,

T_2 = Tension in the slack side of the band,

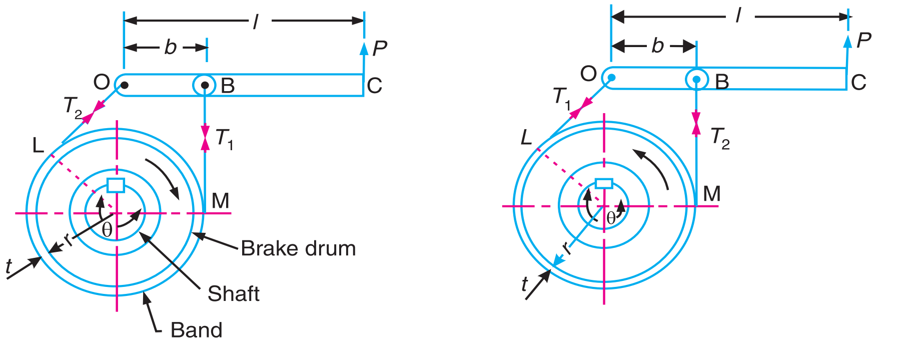
θ = Angle of lap (or embrace) of the band on the drum,

μ = Coefficient of friction between the band and the drum,

r = Radius of the drum,

t = Thickness of the band, and

r_e = Effective radius of the drum $= r + \dfrac{t}{2}$



(*a*) Clockwise rotation of drum. (*b*) Anticlockwise rotation of drum.

Fig. 19.11. Simple band brake.

We know that limiting ratio of the tensions is given by the relation,

$$\frac{T_1}{T_2} = e^{\mu\theta} \quad \text{or} \quad 2.3 \log\left(\frac{T_1}{T_2}\right) = \mu.\theta$$

and braking force on the drum $= T_1 - T_2$

$\therefore$ Braking torque on the drum,

$T_B = (T_1 - T_2)\, r$ . . . (Neglecting thickness of band)

$= (T_1 - T_2)\, r_e$ . . . (Considering thickness of band)

Now considering the equilibrium of the lever OBC. It may be noted that when the drum rotates in the clockwise direction, as shown in Fig. 19.11 (*a*), the end of the band attached to the fulcrum O will be slack with tension T_2 and end of the band attached to B will be tight with tension T_1. On the other hand, when the drum rotates in the anticlockwise direction, as shown in Fig. 19.11 (*b*), the tensions in the band will reverse, *i.e.* the end of the band attached to the fulcrum O will be tight with tension T_1 and the end of the band attached to B will be slack with tension T_2. Now taking moments about the fulcrum O, we have

$P.l = T_1.b$ . . . (For clockwise rotation of the drum)

and $P.l = T_2.b$ . . . (For anticlockwise rotation of the drum)

where l = Length of the lever from the fulcrum (OC), and

b = Perpendicular distance from O to the line of action of T_1 or T_2.

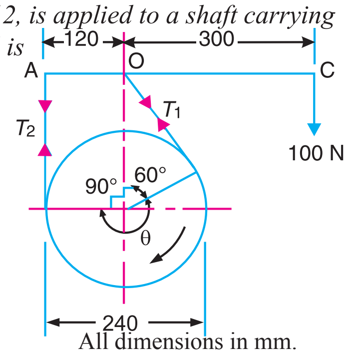
Example 3. *The simple band brake, as shown in Fig. 19.12, is applied to a shaft carrying a flywheel of mass 400 kg. The radius of gyration of the flywheel is 450 mm and runs at 300 r.p.m.*

If the coefficient of friction is 0.2 and the brake drum diameter is 240 mm, find :

1. *the torque applied due to a hand load of 100 N,*

2. *the number of turns of the wheel before it is brought to rest, and*

3. *the time required to bring it to rest, from the moment of the application of the brake.*



All dimensions in mm.

Fig. 19.12

Solution. Given : $m = 400$ kg ; $k = 450$ mm $= 0.45$ m ; $N = 300$ r.p.m. or $\omega = 2\pi \times 300/60 = 31.42$ rad/s ; $\mu = 0.2$; $d = 240$ mm $= 0.24$ m or $r = 0.12$ m

1. Torque applied due to hand load

First of all, let us find the tensions in the tight and slack sides of the band *i.e.* T_1 and T_2 respectively.

From the geometry of the Fig. 19.12, angle of lap of the band on the drum,

$$\theta = 360° - 150° = 210° = 210 \times \frac{\pi}{180} = 3.666 \text{ rad}$$

We know that

$$2.3 \log\left(\frac{T_1}{T_2}\right) = \mu.\theta = 0.2 \times 3.666 = 0.7332$$

$$\log\left(\frac{T_1}{T_2}\right) = \frac{0.7332}{2.3} = 0.3188 \qquad \text{or} \qquad \frac{T_1}{T_2} = 2.08 \qquad \ldots (i)$$

... (Taking antilog of 0.3188)

Taking moments about the fulcrum O,

$$T_2 \times 120 = 100 \times 300 = 30\,000 \qquad \text{or} \qquad T_2 = 30\,000/120 = 250 \text{ N}$$

$$\therefore \qquad T_1 = 2.08\,T_2 = 2.08 \times 250 = 520 \text{ N} \qquad \ldots \text{[From equation } (i)]$$

We know that torque applied,

$$T_B = (T_1 - T_2)\,r = (520 - 250)\,0.12 = 32.4 \text{ N-m} \quad \textbf{Ans.}$$

2. Number of turns of the wheel before it is brought to rest

Let $\qquad n$ = Number of turns of the wheel before it is brought to rest.

We know that kinetic energy of rotation of the drum

$$= \frac{1}{2} \times I.\omega^2 = \frac{1}{2} \times m.k^2.\omega^2 = \frac{1}{2} \times 400(0.45)^2 (31.42)^2 = 40\,000 \text{ N-m}$$

This energy is used to overcome the work done due to the braking torque (T_B).

$$\therefore \qquad 40\,000 = T_B \times 2\pi n = 32.4 \times 2\pi n = 203.6\,n$$

or $$n = 40\,000 / 203.6 = 196.5 \quad \textbf{Ans.}$$

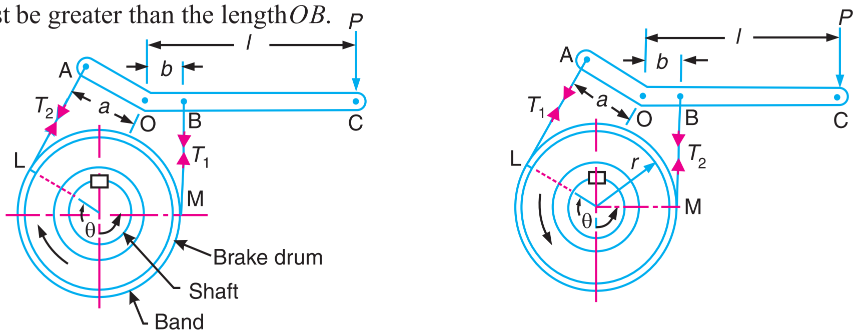
3. Time required to bring the wheel to rest

We know that the time required to bring the wheel to rest

$$= n / N = 196.5 / 300 = 0.655 \text{ min} = 39.3 \text{ s} \quad \textbf{Ans}$$

19.8. Differential Band Brake

In a differential band brake, as shown in Fig. 19.14, the ends of the band are joined at A and B to a lever AOC pivoted on a fixed pin or fulcrum O. It may be noted that for the band to tighten, the length OA must be greater than the length OB.



The braking torque on the drum may be obtained in the similar way as discussed in simple band brake. Now considering the equilibrium of the lever *AOC*. It may be noted that when the drum rotates in the clockwise direction, as shown in Fig. 19.14 (*a*), the end of the band attached to *A* will be slack with tension T_2 and end of the band attached to *B* will be tight with tension T_1. On the other hand, when the drum rotates in the anticlockwise direction, as shown in Fig. 19.14 (*b*), the end of the band attached to *A* will be tight with tension T_1 and end of the band attached to *B* will be slack with tension T_2. Now taking moments about the fulcrum *O*, we have

$$P.l + T_1.b = T_2.a \qquad \text{... (For clockwise rotation of the drum)}$$

or $$P.l = T_2.a - T_1.b \qquad \text{... } (i)$$

and $$P.l + T_2.b = T_1.a \qquad \text{.. (For anticlockwise rotation of the drum)}$$

or $$P.l = T_1.a - T_2.b \qquad \text{... } (ii)$$

We have discussed in block brakes (Art. 19.4), that when the frictional force helps to apply the brake, it is said to be self energizing brake. In case of differential band brake, we see from equations (*i*) and (*ii*) that the moment $T_1.b$ and $T_2.b$ helps in applying the brake (because it adds to the moment $P.l$) for the clockwise and anticlockwise rotation of the drum respectively

We have also discussed that when the force *P* is negative or zero, then brake is self locking. Thus for differential band brake and for clockwise rotation of the drum, the condition for self locking is

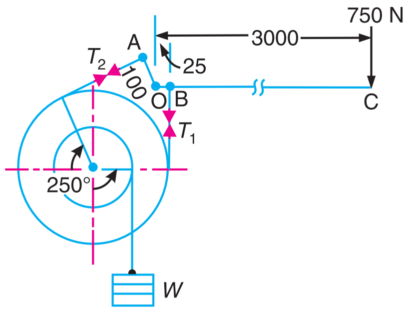
$$T_2.a \le T_1.b \qquad \text{or} \qquad T_2 / T_1 \le b / a$$

and for anticlockwise rotation of the drum, the condition for self locking is

$$T_1.a \le T_2.b \qquad \text{or} \qquad T_1 / T_2 \le b / a$$

Example 4. *In a winch, the rope supports a load W and is wound round a barrel 450 mm diameter. A differential band brake acts on a drum 800 mm diameter which is keyed to the same shaft as the barrel. The two ends of the bands are attached to pins on opposite sides of the fulcrum of the brake lever and at distances of 25 mm and 100 mm from the fulcrum. The angle of lap of the brake band is 250° and the coefficient of friction is 0.25. What is the maximum load W which can be supported by the brake when a force of 750 N is applied to the lever at a distance of 3000 mm from the fulcrum ?*

Solution. Given : $D = 450$ mm or $R = 225$ mm ; $d = 800$ mm or $r = 400$ mm ; $OB = 25$ mm ; $OA = 100$ mm ; $\theta = 250° = 250 \times \pi/180 = 4.364$ rad ; $\mu = 0.25$; $P = 750$ N ; $l = OC = 3000$ mm



All dimensions in mm.
Fig. 19.15

Since *OA* is greater than *OB*, therefore the operating force ($P = 750$ N) will act downwards.

First of all, let us consider that the drum rotates in clockwise direction.

We know that when the drum rotates in clockwise direction, the end of band attached to *A* will be slack with tension T_2 and the end of the band attached to *B* will be tight with tension T_1, as shown in Fig. 19.15. Now let us find out the values of tensions T_1 and T_2. We know that

$$2.3 \log\left(\frac{T_1}{T_2}\right) = \mu.\theta = 0.25 \times 4.364 = 1.091$$

$$\therefore \quad \log\left(\frac{T_1}{T_2}\right) = \frac{1.091}{2.3} = 0.4743 \text{ or}$$

T1 = 2.98 T2... (i)

Now taking moments about the fulcrum O, 750 × 3000 + T1 × 25 = T2 × 100

or $T_2 \times 100 - 2.98\, T_2 \times 25 = 2250 \times 10^3 \qquad \text{... } (\because T_1 = 2.98\, T_2)$

$$25.5\, T_2 = 2250 \times 10^3 \quad \text{or} \quad T_2 = 2250 \times 10^3/25.5 = 88 \times 10^3 \text{ N}$$

and $$T_1 = 2.98 T_2 = 2.98 \times 88 \times 10^3 = 262 \times 10^3 \text{ N}$$

We know that braking torque,

$$T_B = (T_1 - T_2)\, r$$
$$= (262 \times 10^3 - 88 \times 10^3)\, 400 = 69.6 \times 10^6 \text{ N-mm} \qquad \text{...}(i)$$

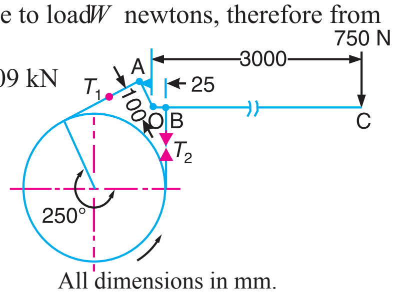
(∵

and the torque due to load W newtons,

$$T_W = W.R = W \times 225 = 225\ W \text{ N-mm} \quad \text{... } (ii)$$

Since the braking torque must be equal to the torque due to load W newtons, therefore from equations (*i*) and (*ii*),

$$W = 69.6 \times 10^6/225 = 309 \times 10^3 \text{ N} = 309 \text{ kN}$$



All dimensions in mm.

Fig. 19.16

Now let us consider that the drum rotates in anticlockwise direction. We know that when the drum rotates in anticlockwise direction, the end of the band attached to A will be tight with tension T_1 and end of the band attached to B will be slack with tension T_2, as shown in Fig. 19.16. The ratio of tensions T_1 and T_2 will be same as calculated above, *i.e.*

$$\frac{T_1}{T_2} = 2.98 \text{ or } T_1 = 2.98\ T_2$$

Now taking moments about the fulcrum O,

$$750 \times 3000 + T_2 \times 25 = T_1 \times 100$$

or $2.98\ T_2 \times 100 - T_2 \times 25 = 2250 \times 10^3$ ... ($\because T_1 = 2.98\ T_2$)

$$273\ T_2 = 2250 \times 10^3 \quad \text{or} \quad T_2 = 2250 \times 10^3/273 = 8242 \text{ N}$$

and $T_1 = 2.98\ T_2 = 2.98 \times 8242 = 24\ 561$ N

∴ Braking torque, $T_B = (T_1 \times T_2)\ r$

$$= (24\ 561 - 8242)400 = 6.53 \times 10^6 \text{ N-mm} \quad \text{...}(iii)$$

From equations (*ii*) and (*iii*),

$$W = 6.53 \times 10^6/225 = 29 \times 10^3 \text{ N} = 29 \text{ kN}$$

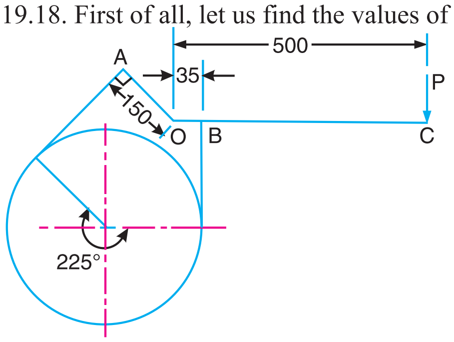
From above, we see that the maximum load (W) that can be supported by the brake is 309 kN, when the drum rotates in clockwise direction. **Ans.**

Example 5. *A differential band brake, as shown in Fig. 19.17, has an angle of contact of 225°. The band has a compressed woven lining and bears against a cast iron drum of 350 mm diameter. The brake is to sustain a torque of 350 N-m and the coefficient of friction between the band and the drum is 0.3. Find :* **1.** *The necessary force (P) for the clockwise and anticlockwise rotation of the drum; and* **2.** *The value of 'OA' for the brake to be self locking, when the drum rotates clockwise.*

Solution. Given: $\theta = 225° = 225 \times \pi/180 = 3.93$ rad ; $d = 350$ mm or $r = 175$ mm ; $T = 350$ N-m $= 350 \times 10^3$ N-mm

1. *Necessary force (P) for the clockwise and anticlockwise rotation of the drum*

When the drum rotates in the clockwise direction, the end of the band attached to A will be slack with tension T_2 and the end of the band attached to B will be tight with tension T_1, as shown in Fig. 19.18. First of all, let us find the values of tensions T_1 and T_2.



All dimensions in mm.

Fig. 19.17

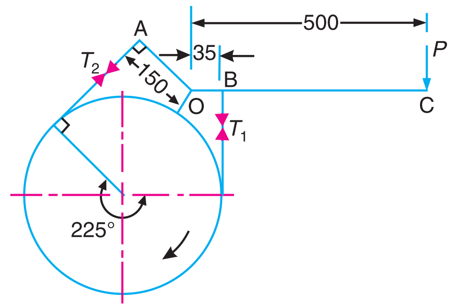


Fig. 19.18

We know that $2.3 \log\left(\frac{T_1}{T_2}\right) = \mu.\theta = 0.3 \times 3.93 = 1.179$

$$\therefore \quad \log\left(\frac{T_1}{T_2}\right) = \frac{1.179}{2.3} = 0.5126 \quad \text{or}$$

$$\frac{T_1}{T_2} = 3.255 \qquad \text{... (Taking antilog of 0.5126) ... } (i)$$

and braking torque (T_B),

$$350 \times 10^3 = (T_1 - T_2)r = (T_1 - T_2)\ 175$$

$$\therefore \qquad T_1 - T_2 = 350 \times 10^3/175 = 2000 \text{ N} \qquad \text{... } (ii)$$

From equations ***(i)*** and ***(ii)***, we find that

$$T_1 = 2887 \text{ N ; and } T_2 = 887 \text{ N}$$

Now taking moments about the fulcrum O, we have

$$P \times 500 = T_2 \times 150 - T_1 \times 35 = 887 \times 150 - 2887 \times 35 = 32 \times 10^3$$

$$\therefore \qquad P = 32 \times 10^3/500 = 64 \text{ N} \quad \textbf{Ans.}$$

When the drum rotates in the anticlockwise direction, the end of the band attached to A will be tight with tension T_1 and end of the band attached to B will be slack with tension T_2, as shown in Fig. 19.19. Taking moments about the fulcrum O, we have

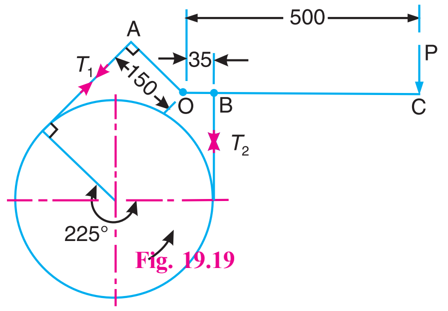


Fig. 19.19

$$P \times 500 = T_1 \times 150 - T_2 \times 35$$
$$= 2887 \times 150 - 887 \times 35$$
$$= 402 \times 10^3$$
$$P = 402 \times 10^3/500 = 804 \text{ N} \quad \textbf{Ans.}$$

2. Value of 'OA' for the brake to be self locking, when the drum rotates clockwise

The clockwise rotation of the drum is shown in Fig 19.18.

For clockwise rotation of the drum, we know that

$$P \times 500 = T_2 \times OA - T_1 \times OB$$

For the brake to be self locking, P must be equal to zero. Therefore

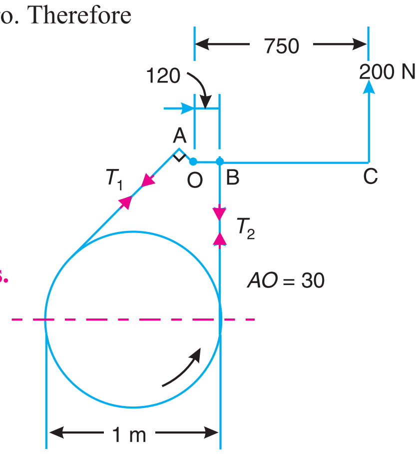
$$T_2 \times OA = T_1 \times OB$$

Taking moments about O,

$$200 \times 750 + T_1 \times 30 = T_2 \times 120$$

$$12\ T_2 - 3T_1 = 15\ 000 \qquad \text{. . . } (i)$$

We know that $OA = \dfrac{T_1 \times OB}{T_2} = \dfrac{2887 \times 35}{887} = 114 \text{ mm}$ **Ans.**



All dimensions in mm

Fig. 19.23

$$\frac{T_1}{T_2} = \left(\frac{1 + \mu \tan\theta}{1 - \mu \tan\theta}\right)^n$$

$$= \left(\frac{1 + 0.25 \tan 7.5^\circ}{1 - 0.25 \tan 7.5^\circ}\right)^{14} = \left(\frac{1 + 0.25 \times 0.1317}{1 - .025 \times 0.1317}\right)^{14}$$

$$= (1.068)^{14} = 2.512 \text{ . . . } (ii)$$

From equations ***(i)*** and ***(ii)***,

$$T_1 = 8440 \text{ N, and } T_2 = 3360 \text{ N}$$

We know that maximum braking torque,

$$T_B = (T_1 - T_2)\,r = (8440 - 3360)0.5 = 2540 \text{ N-m} \quad \textbf{Ans.}$$

2. Angular retardation of the drum

Let $\qquad \alpha$ = Angular retardation of the drum.

We know that braking torque (T_B),

$$2540 = I.\alpha = m.k^2.\alpha = 2000(0.5)^2\,\alpha = 500\,\alpha$$

$$\therefore \qquad \alpha = 2540/500 = 5.08 \text{ rad/s}^2 \quad \textbf{Ans.}$$

19.10. Internal Expanding Brake

3. *Time taken by the system to come to rest*

Let t = Required time.

Since the system is to come to rest from the rated speed of 360 r.p.m., therefore

Initial angular speed, $\omega_1 = 2\pi \times 360 / 60 = 37.7$ rad/s

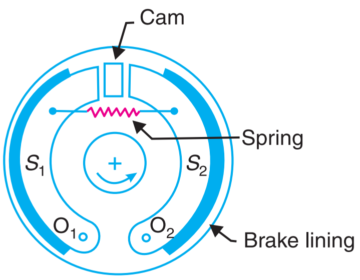
and final angular speed, $\omega_2 = 0$

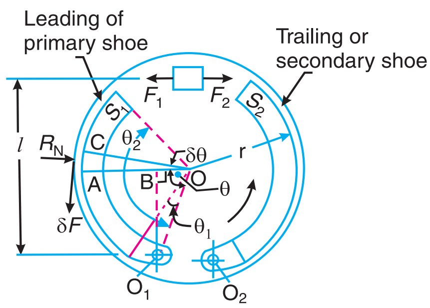
We know that $\omega_2 = \omega_1 - \alpha.t$ . . . (– ve sign due to retardation)

$\therefore$ $t = \omega_1 / \alpha = 37.7 / 5.08 = 7.42$ s **Ans.**

19.10. Internal Expanding Brake

An internal expanding brake consists of two shoes S_1 and S_2 as shown in Fig. 19.24. The outer surface of the shoes are lined with some friction material (usually with Ferodo) to increase the coefficient of friction and to prevent wearing away of the metal. Each shoe is pivoted at one end about a fixed fulcrum O_1 and O_2 and made to contact a cam at the other end. When the cam rotates, the shoes are pushed outwards against the rim of the drum. The friction between the shoes and the drum produces the braking torque and hence reduces the speed of the drum. The shoes are normally held in off position by a spring as shown in Fig. 19.24. The drum encloses the entire mechanism to keep out dust and moisture. This type of brake is commonly used in motor cars and light trucks.





We shall now consider the forces acting on such a brake, when the drum rotates in the anticlockwise direction as shown in Fig. It may be noted that for the anticlockwise direction, the left hand shoe is known as ***leading*** or ***primary shoe*** while the right hand shoe is known as ***trailing*** or ***secondary shoe.***

Let r = Internal radius of the wheel rim,

b = Width of the brake lining,

p_1 = Maximum intensity of normal pressure,

p_N = Normal pressure,

F_1 = Force exerted by the cam on the leading shoe, and

F_2 = Force exerted by the cam on the trailing shoe.

Consider a small element of the brake lining AC subtending an angle $\delta\theta$ at the centre. Let OA makes an angle θ with OO_1 as shown in Fig. 19.25. It is assumed that the pressure distribution on the shoe is nearly uniform, however the friction lining wears out more at the free end. Since the shoe turns about O_1, therefore the rate of wear of the shoe lining at A will be proportional to the radial displacement of that point.

The rate of wear of the shoe lining varies directly as the perpendicular distance from O_1 to OA, *i.e.* O_1B.

From the geometry of the figure,

$$O_1B = OO_1 \sin\theta$$

and normal pressure at A,

$$p_N \propto \sin\theta \quad \text{or} \quad p_N = p_1 \sin\theta$$

$\therefore$ Normal force acting on the element,

$$\delta R_N = \text{Normal pressure} \times \text{Area of the element}$$

$$= p_N (b.r.\delta\theta) = p_1 \sin\theta (b.r.\delta\theta)$$

and braking or friction force on the element,

$$\delta F = \mu \times \delta R_N = \mu . p_1 \sin\theta (b.r.\delta\theta)$$

$\therefore$ Braking torque due to the element about O,

$$\delta T_B = \delta F \times r = \mu . p_1 \sin\theta (b.r.\delta\theta) r = \mu . p_1 b\, r^2 (\sin\theta . \delta\theta)$$

and total braking torque about O for whole of one shoe,

$$T_B = \mu\, p_1 b\, r^2 \int_{\theta_1}^{\theta_2} \sin\theta\, d\theta = \mu\, p_1 b\, r^2 [-\cos\theta]_{\theta_1}^{\theta_2}$$

$$= \mu\, p_1 b r^2 (\cos\theta_1 - \cos\theta_2)$$

Moment of normal force δR_N of the element about the fulcrum O_1,

$$\delta M_N = \delta R_N \times O_1 B = \delta R_N (OO_1 \sin\theta)$$

$$= p_1 \sin\theta (b.r.\delta\theta)(OO_1 \sin\theta) = p_1 \sin^2\theta (b.r.\delta\theta) OO_1$$

$\therefore$ Total moment of normal forces about the fulcrum O_1,

$$M_N = \int_{\theta_1}^{\theta_2} p_1 \sin^2\theta (b.r.\delta\theta) OO_1 = p_1 . b.r.OO_1 \int_{\theta_1}^{\theta_2} \sin^2\theta\, d\theta$$

$$= p_1 . b.r.OO_1 \int_{\theta_1}^{\theta_2} \frac{1}{2}(1 - \cos 2\theta) d\theta \qquad \ldots \left[\because \sin^2\theta = \frac{1}{2}(1 - \cos 2\theta)\right]$$

$$= \frac{1}{2} p_1 . b.r.OO_1 \left[\theta - \frac{\sin 2\theta}{2}\right]_{\theta_1}^{\theta_2}$$

$$= \frac{1}{2} p_1 . b.r.OO_1 \left[\theta_2 - \frac{\sin 2\theta_2}{2} - \theta_1 + \frac{\sin 2\theta_1}{2}\right]$$

$$= \frac{1}{2} p_1 . b.r.OO_1 \left[(\theta_2 - \theta_1) + \frac{1}{2}(\sin 2\theta_1 - \sin 2\theta_2)\right]$$

Moment of frictional force δF about the fulcrum O_1,

$$\delta M_F = \delta F \times AB = \delta F (r - OO_1 \cos\theta) \qquad \ldots (\because AB = r - OO_1 \cos\theta)$$

$$= \mu\, p_1 \sin\theta (b.r.\delta\theta)(r - OO_1 \cos\theta)$$

$$= \mu . p_1 . b.r (r \sin\theta - OO_1 \sin\theta \cos\theta) \delta\theta$$

$$= \mu . p_1 . b.r \left(r \sin\theta - \frac{OO_1}{2} \sin 2\theta\right) \delta\theta \qquad \ldots (\because 2\sin\theta\cos\theta = \sin 2\theta)$$

$\therefore$ Total moment of frictional force about the fulcrum O_1,

$$M_F = \mu\, p_1 b\, r \int_{\theta_1}^{\theta_2} \left(r \sin\theta - \frac{OO_1}{2} \sin 2\theta\right) d\theta$$

$$= \mu\, p_1 b\, r \left[-r\cos\theta + \frac{OO_1}{4} \cos 2\theta\right]_{\theta_1}^{\theta_2}$$

$$= \mu\, p_1 b\, r \left[-r\cos\theta_2 + \frac{OO_1}{4}\cos 2\theta_2 + r\cos\theta_1 - \frac{OO_1}{4}\cos 2\theta_1\right]$$

$$= \mu\, p_1 b\, r \left[r(\cos\theta_1 - \cos\theta_2) + \frac{OO_1}{4}(\cos 2\theta_2 - \cos 2\theta_1)\right]$$

Now for leading shoe, taking moments about the fulcrum O_1,

$$F_1 \times l = M_N - M_F$$

and for trailing shoe, taking moments about the fulcrum O_2,

$$F_2 \times l = M_N + M_F$$

Note : If $M_F > M_N$, then the brake becomes self locking.

Example 6. *The arrangement of an internal expanding friction brake, in which the brake shoe is pivoted at 'C' is shown in Fig. 19.26. The distance 'CO' is 75 mm, O being the centre of the drum. The internal radius of the brake drum is 100 mm. The friction lining extends over an arc AB, such that the angle AOC is 135° and angle BOC is 45°. The brake is applied by means of a force at Q, perpendicular to the line CQ, the distance CQ being 150 mm.*

The local rate of wear on the lining may be taken as proportional to the normal pressure on an element at an angle of 'θ' with OC and may be taken as equal to $p_1 \sin\theta$, where p_1 is the maximum intensity of normal pressure.

The coefficient of friction may be taken as 0.4 and the braking torque required is 21 N-m. Calculate the force Q required to operate the brake when **1.** *The drum rotates clockwise, and* **2.** *The drum rotates anticlockwise.*

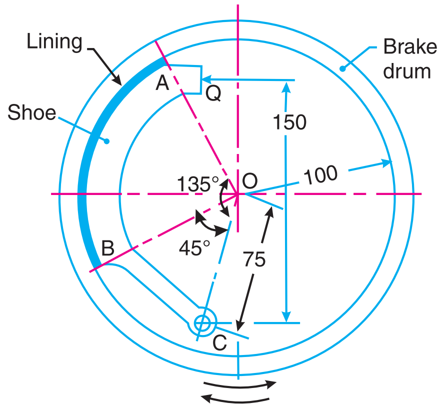


Fig. 19.26

Solution. Given : $OC = 75$ mm ; $r = 100$ mm ;

$\theta_2 = 135° = 135 \times \pi/180 = 2.356$ rad ; $\theta_1 = 45° = 45 \times \pi/180 = 0.786$ rad ; $l = 150$ mm ; $\mu = 0.4$; $T_B = 21$ N-m $= 21 \times 10^3$ N-mm

1. *Force 'Q' required to operate the brake when drum rotates clockwise*

We know that total braking torque due to shoe (T_B),

$$21 \times 10^3 = \mu . p_1 . b . r^2 (\cos\theta_1 - \cos\theta_2)$$

$$= 0.4 \times p_1 \times b (100)^2 \, (\cos 45° - \cos 135°) = 5656 \; p_1 . b$$

$$\therefore \quad p_1 . b = 21 \times 10^3 / 5656 = 3.7$$

Total moment of normal forces about the fulcrum C,

$$M_N = \frac{1}{2} p_1 . b . r . OC \left[(\theta_2 - \theta_1) + \frac{1}{2} (\sin 2\theta_1 - \sin 2\theta_2) \right]$$

$$= \frac{1}{2} \times 3.7 \times 100 \times 75 \left[(2.356 - 0.786) + \frac{1}{2} (\sin 90° - \sin 270°) \right]$$

$$= 13\,875\,(1.57 + 1) = 35\,660 \text{ N-mm}$$

and total moment of friction force about the fulcrum C,

$$M_F = \mu . p_1 . b . r \left[r (\cos\theta_1 - \cos\theta_2) + \frac{OC}{4} (\cos 2\theta_2 - \cos 2\theta_1) \right]$$

$$= 0.4 \times 3.7 \times 100 \left[100 (\cos 45° - \cos 135°) + \frac{75}{4} (\cos 270° - \cos 90°) \right]$$

$$= 148 \times 141.4 = 20\,930 \text{ N-mm}$$

Taking moments about the fulcrum C, we have

$$Q \times 150 = M_N + M_F = 35\,660 + 20\,930 = 56\,590$$

$$\therefore \quad Q = 56\,590 / 150 = 377 \text{ N} \quad \textbf{Ans.}$$

2. *Force 'Q' required to operate the brake when drum rotates anticlockwise*

Taking moments about the fulcrum C, we have

$$Q \times 150 = M_N - M_F = 35\,660 - 20\,930 = 14\,730$$

$$\therefore \quad Q = 14\,730/150 = 98.2 \text{ N} \quad \textbf{Ans.}$$

19.11. Braking of a Vehicle

In a four wheeled moving vehicle, the brakes may be applied to

1. the rear wheels only,

2. the front wheels only, and

3. all the four wheels.

Example 7. *A car moving on a level road at a speed 50 km/h has a wheel base 2.8 metres, distance of C.G. from ground level 600 mm, and the distance of C.G. from rear wheels 1.2 metres. Find the distance travelled by the car before coming to rest when brakes are applied,*

***1.** to the rear wheels, **2.** to the front wheels, and **3.** to all the four wheels.*

The coefficient of friction between the tyres and the road may be taken as 0.6.

Solution. Given : $u = 50$ km/h $= 13.89$ m/s ; $L = 2.8$ m ; $h = 600$ mm $= 0.6$ m ; $x = 1.2$ m ; $\mu = 0.6$

Let s = Distance travelled by the car before coming to rest.

1. ***When brakes are applied to the rear wheels***

Since the vehicle moves on a level road, therefore retardation of the car,

$$a = \frac{\mu.g(L-x)}{L+\mu.h} = \frac{0.6 \times 9.81(2.8-1.2)}{2.8+0.6 \times 0.6} = 2.98 \text{ m/s}^2$$

We know that for uniform retardation,

$$s = \frac{u^2}{2a} = \frac{(13.89)^2}{2 \times 2.98} = 32.4 \text{ m} \quad \textbf{Ans.}$$

2. ***When brakes are applied to the front wheels***

Since the vehicle moves on a level road, therefore retardation of the car,

$$a = \frac{\mu.g.x}{L-\mu.h} = \frac{0.6 \times 9.18 \times 1.2}{2.8-0.6 \times 0.6} = 2.9 \text{ m/s}^2$$

We know that for uniform retardation,

$$s = \frac{u^2}{2a} = \frac{(13.89)^2}{2 \times 2.9} = 33.26 \text{ m} \quad \textbf{Ans.}$$

3. ***When the brakes are applied to all the four wheels***

Since the vehicle moves on a level road, therefore retardation of the car,

$$a = g.\mu = 9.81 \times 0.6 = 5.886 \text{ m/s}^2$$

We know that for uniform retardation,

$$s = \frac{u^2}{2a} = \frac{(13.89)^2}{2 \times 5.886} = 16.4 \text{ m} \quad \textbf{Ans.}$$

19.12. Dynamometer

A dynamometer is a brake but in addition it has a device to measure the frictional resistance. Knowing the frictional resistance, we may obtain the torque transmitted and hence the power of the engine.

19.13. Types of Dynamometers

Following are the two types of dynamometers, used for measuring the brake power of an engine.

1. Absorption dynamometers, and

2. Transmission dynamometers.

In the *absorption **dynamometers,*** the entire energy or power produced by the engine is absorbed by the friction resistances of the brake and is transformed into heat, during the process of measurement. But in the ***transmission dynamometers,*** the energy is not wasted in friction but is used for doing work.

19.14. Classification of Absorption Dynamometers

The following two types of absorption dynamometers are important from the subject point of view :

1. Prony brake dynamometer, and **2.** Rope brake dynamometer.

19.15. Prony Brake Dynamometer

A simplest form of an absorption type dynamometer is a prony brake dynamometer, as shown in Fig. 19.31. It consists of two wooden blocks placed around a pulley fixed to the shaft of an engine whose power is required to be measured. The blocks are clamped by means of two bolts and nuts, as shown in Fig. 19.31. A helical spring is provided between the nut and the upper block to adjust the pressure on the pulley to control its speed. The upper block has a long lever attached to it and carries a weight W at its outer end. A counter weight is placed at the other end of the lever which balances the brake when unloaded. Two stops S, S are provided to limit the motion of the lever.

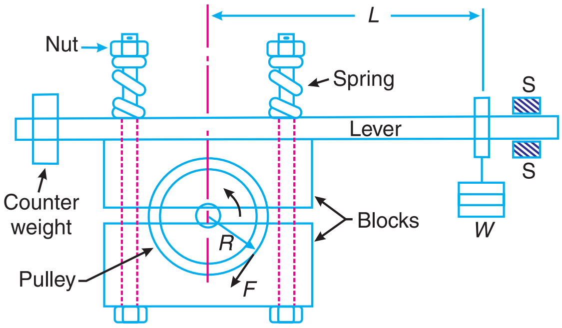


Fig. 19.31. Prony brake dynamometer.

When the brake is to be put in operation, the long end of the lever is loaded with suitable weights W and the nuts are tightened until the engine shaft runs at a constant speed and the lever is in horizontal position. Under these conditions, the moment due to the weight W must balance the moment of the frictional resistance between the blocks and the pulley.

Let W = Weight at the outer end of the lever in newtons,

L = Horizontal distance of the weight W from the centre of the pulley in metres,

F = Frictional resistance between the blocks and the pulley in newtons,

R = Radius of the pulley in metres, and N = Speed of the shaft in r.p.m.

We know that the moment of the frictional resistance or torque on the shaft,

$$T = W.L = F.R \text{ N-m}$$

Work done in one revolution = Torque × Angle turned in radians = $T \times 2\pi$ N-m

$\therefore$ Work done per minute $= T \times 2\pi\, N$ N-m

We know that brake power of the engine,

$$B.P. = \frac{\text{Work done per min.}}{60} = \frac{T \times 2\pi N}{60} = \frac{W.L \times 2\pi\, N}{60} \text{ watts}$$

19.16. Rope Brake Dynamometer

It is another form of absorption type dynamometer which is most commonly used for measuring the brake power of the engine. It consists of one, two or more ropes wound around the flywheel or rim of a pulley fixed rigidly to the shaft of an engine. The upper end of the ropes is attached to a spring balance while the lower end of the ropes is kept in position by applying a dead weight as shown in Fig. 19.32. In order to prevent the slipping of the rope over the flywheel, wooden blocks are placed at intervals around the circumference of the flywheel.

In the operation of the brake, the engine is made to run at a constant speed. The frictional torque, due to the rope, must be equal to the torque being transmitted by the engine.

Let W = Dead load in newtons,

S = Spring balance reading in newtons,

D = Diameter of the wheel in metres,

d = diameter of rope in metres, and

N = Speed of the engine shaft in r.p.m.

$\therefore$ Net load on the brake

$$= (W - S) \text{ N}$$

We know that distance moved in one revolution

$$= \pi(D + d) \text{ m}$$

$\therefore$ Work done per revolution

$$= (W - S)\,\pi\,(D + d) \text{ N-m}$$

and work done per minute

$$= (W - S)\,\pi\,(D + d)\, N \text{ N-m}$$

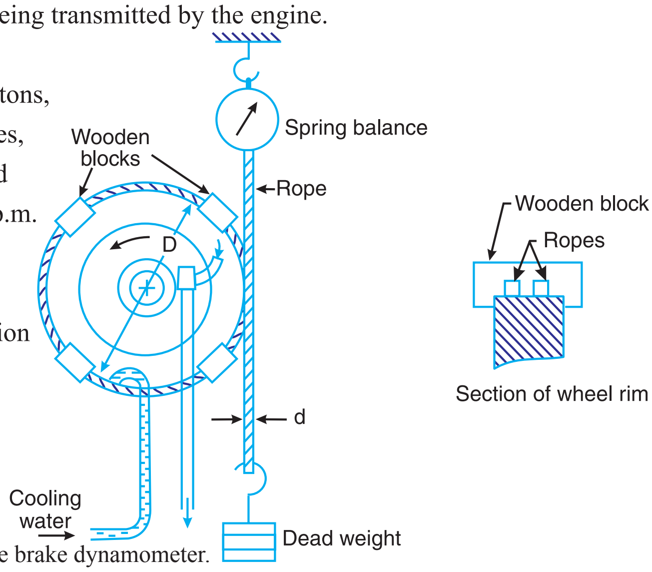


Fig. 19.32. Rope brake dynamometer.

$\therefore$ Brake power of the engine,

$$\text{B.P} = \frac{\text{Work done per min}}{60} = \frac{(W - S)\,\pi\,(D + d) N}{60} \text{ watts}$$

If the diameter of the rope (d) is neglected, then brake power of the engine,

$$\text{B.P.} = \frac{(W - S)\,\pi\, D\, N}{60} \text{ watts}$$

UNIT-IV BALANCING OF ROTATING MASSES

The process of providing the second mass in order to counteract the effect of the centrifugal force of the first mass, is called ***balancing of rotating masses*.**

21.3. Balancing of a Single Rotating Mass By a Single Mass Rotating in the Same Plane

Consider a disturbing mass m_1 attached to a shaft rotating at ω rad/s as shown in Fig. 21.1. Let r_1 be the radius of rotation of the mass m_1.

.

We know that the centrifugal force exerted by the mass m_1 on the shaft,

$$F_{C1} = m_1 \cdot \omega^2 \cdot r_1 \qquad \ldots (i)$$

This centrifugal force acts radially outwards and thus produces bending moment on the shaft.

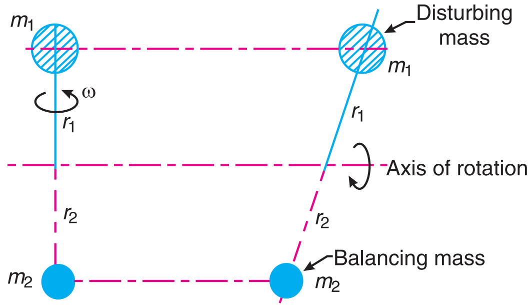


Fig. 21.1. Balancing of a single rotating mass by a single mass rotating in the same plane.

Let $\qquad r_2$ = Radius of rotation of the balancing mass m_2.

∴ Centrifugal force due to mass m_2,

$$F_{C2} = m_2 \cdot \omega^2 \cdot r_2 \qquad \ldots (ii)$$

Equating equations **(*i*)** and **(*ii*),**

$$m_1 . \omega^2 \cdot r_1 = m_2 \cdot \omega^2 \cdot r_2 \quad \text{or} \quad m_1 \cdot r_1 = m_2 \cdot r_2$$

Discuss how a single revolving mass is balanced by two masses revolving in different planes ?

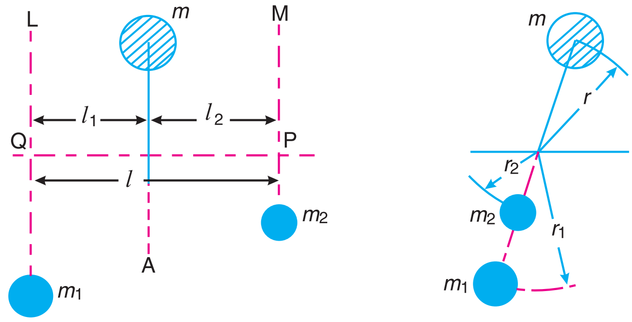
21.4. Balancing of a Single Rotating Mass By Two Masses Rotating in Different Planes

Two balancing masses are placed in two different planes, parallel to the plane of rotation of the disturbing mass, in such a way that they satisfy the following two conditions of equilibrium.

1. The net dynamic force acting on the shaft is equal to zero. This requires that the line of action of three centrifugal forces must be the same. In other words, the centre of the masses of the system must lie on the axis of rotation. This is the condition for S***tatic balancing*.**
2. The net couple due to the dynamic forces acting on the shaft is equal to zero. In other words, the algebraic sum of the moments about any point in the plane must be zero. The conditions **(1)** and **(2)** together give ***dynamic balancing*.**

1. ***When the plane of the disturbing mass lies in between the planes of the two balancing masses***

Consider a disturbing mass m lying in a plane A to be balanced by two rotating masses m_1 and m_2 lying in two different planes L and M as shown in Fig. 21.2. Let r, r_1 and r_2 be the radii of rotation of the masses in planes A, L and M respectively.



We know that the centrifugal force exerted by the mass m in the plane A,

$$F_C = m \cdot \omega^2 \cdot r$$

Similarly, the centrifugal force exerted by the mass m_1 in the plane L,

$$F_{C1} = m_1 \cdot \omega^2 \cdot r_1$$

and, the centrifugal force exerted by the mass m_2 in the plane M,

$$F_{C2} = m_2 \cdot \omega^2 \cdot r_2$$

Since the net force acting on the shaft must be equal to zero, therefore the centrifugal force on the disturbing mass must be equal to the sum of the centrifugal forces on the balancing masses, therefore

$$F_C \quad F_{C1} \quad F_{C2}$$

$$\therefore \qquad m \cdot r = m_1 \cdot r_1 + m_2 \cdot r_2 \qquad \text{. . . } (i)$$

Now in order to find the magnitude of balancing force in the plane L (or the dynamic force at the bearing Q of a shaft), take moments about P which is the point of intersection of the plane M and the axis of rotation. Therefore

$$F_{C1} \times l = F_C \times l_2$$

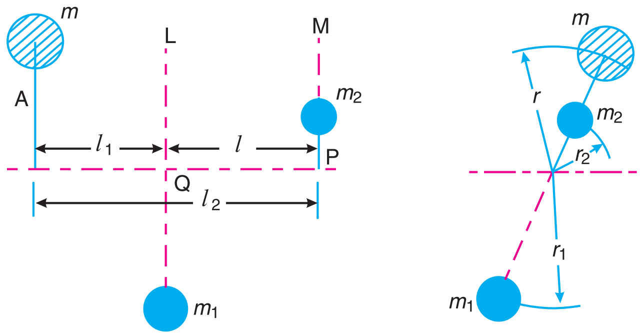
$$\therefore \qquad m_1 \cdot r_1 \cdot l = m \cdot r \cdot l_2 \qquad \text{. . . } (ii)$$

Similarly, in order to find the balancing force in plane M (or the dynamic force at the bearing P of a shaft), take moments about Q which is the point of intersection of the plane L and the axis of rotation. Therefore

$$F_{C2} \times l = F_C \times l_1$$

$$\therefore \qquad m_2 \cdot r_2 \cdot l = m \cdot r \cdot l_1 \qquad \text{. . . } (iii)$$

2. *When the plane of the disturbing mass lies on one end of the planes of the balancing masses*



In this case, the mass m lies in the plane A and the balancing masses lie in the planes L and M, the following conditions must be satisfied in order to balance the system, *i.e.*

$$F_C + F_{C2} = F_{C1}$$

$$\therefore \quad m \cdot r + m_2 \cdot r_2 = m_1 . r_1 \quad \ldots (iv)$$

Now, to find the balancing force in the plane L (or the dynamic force at the bearing Q of a shaft), take moments about P which is the point of intersection of the plane M and the axis of rotation. Therefore

$$F_{C1} \times l = F_C \times l_2$$

$$\therefore \quad m_1 \cdot r_1 \cdot l = m \cdot r \cdot l_2 \quad \ldots (v)$$

. . . [Same as equation (*ii*)]

Similarly, to find the balancing force in the plane M (or the dynamic force at the bearing P of a shaft), take moments about Q which is the point of intersection of the plane L and the axis of rotation. Therefore

$$F_{C2} \times l = F_C \times l_1$$

$$m_2 \cdot r_2 \cdot l = m \cdot r \cdot l_1$$

21.5. Balancing of Several Masses Rotating in the Same Plane

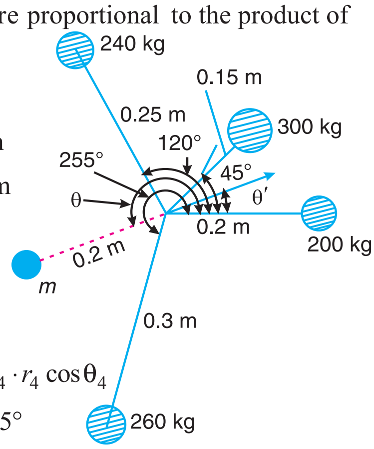
Example 1. *Four masses m_1, m_2, m_3 and m_4 are 200 kg, 300 kg, 240 kg and 260 kg respectively. The corresponding radii of rotation are 0.2 m, 0.15 m, 0.25 m and 0.3 m respectively and the angles between successive masses are 45°, 75° and 135°. Find the position and magnitude of the balance mass required, if its radius of rotation is 0.2 m.*

Solution. Given : $m_1 = 200$ kg ; $m_2 = 300$ kg ; $m_3 = 240$ kg ; $m_4 = 260$ kg ; $r_1 = 0.2$ m ; $r_2 = 0.15$ m ; $r_3 = 0.25$ m ; $r_4 = 0.3$ m ; $\theta_1 = 0°$; $\theta_2 = 45°$; $\theta_3 = 45° + 75° = 120°$; $\theta_4 = 45° + 75° + 135° = 255°$; $r = 0.2$ m

Let m = Balancing mass, and

θ = The angle which the balancing mass makes with m_1.

Since the magnitude of centrifugal forces are proportional to the product of each mass and its radius, therefore



$$m_1 \cdot r_1 = 200 \times 0.2 = 40 \text{ kg-m}$$

$$m_2 \cdot r_2 = 300 \times 0.15 = 45 \text{ kg-m}$$

$$m_3 \cdot r_3 = 240 \times 0.25 = 60 \text{ kg-m}$$

$$m_4 \cdot r_4 = 260 \times 0.3 = 78 \text{ kg-m}$$

1. *Analytical method*

$$\Sigma H = m_1 \cdot r_1 \cos\theta_1 + m_2 \cdot r_2 \cos\theta_2 + m_3 \cdot r_3 \cos\theta_3 + m_4 \cdot r_4 \cos\theta_4$$

$$= 40\cos 0° + 45\cos 45° + 60\cos 120° + 78\cos 255°$$

$$= 40 + 31.8 - 30 - 20.2 = 21.6 \text{ kg-m}$$

Now resolving vertically,

$$\Sigma V = m_1 \cdot r_1 \sin\theta_1 + m_2 \cdot r_2 \sin\theta_2 + m_3 \cdot r_3 \sin\theta_3 + m_4 \cdot r_4 \sin\theta_4$$

$$= 40\sin 0° + 45\sin 45° + 60\sin 120° + 78\sin 255°$$

$$= 0 + 31.8 + 52 - 75.3 = 8.5 \text{ kg-m}$$

$$\text{Resultant, } R = \sqrt{(\Sigma H)^2 + (\Sigma V)^2} = \sqrt{(21.6)^2 + (8.5)^2} = 23.2 \text{ kg-m}$$

We know that

$$m \cdot r = R = 23.2 \quad \text{or} \quad m = 23.2 / r = 23.2 / 0.2 = 116 \text{ kg} \quad \textbf{Ans.}$$

and $\tan\theta' = \Sigma V / \Sigma H = 8.5 / 21.6 = 0.3935$ or $\theta' = 21.48°$

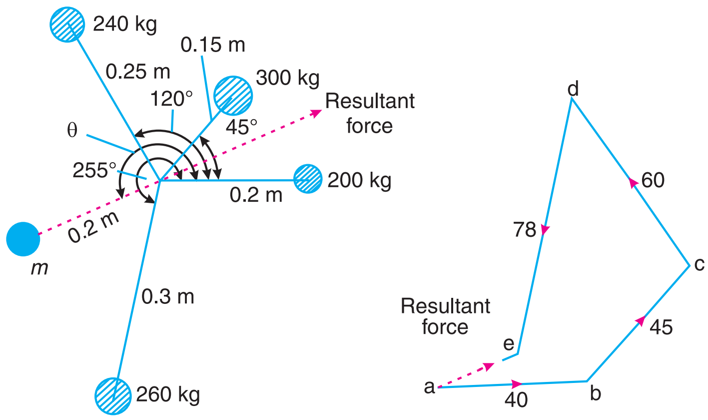
Since θ' is the angle of the resultant R from the horizontal mass of 200 kg, therefore the angle of the balancing mass from the horizontal mass of 200 kg

$$\theta = 180° + 21.48° = 201.48° \quad \textbf{Ans.}$$

2. *Graphical method*

1. First of all, draw the space diagram showing the positions of all the given masses as shown in Fig 21.6 (*a*).
2. Since the centrifugal force of each mass is proportional to the product of the mass and radius, therefore

$$m_1.r_1 = 200 \times 0.2 = 40 \text{ kg-m} \;\&\; m_2.r_2 = 300 \times 0.15 = 45 \text{ kg-m}$$



(*a*) Space diagram. (*b*) Vector diagram

Fig. 21.6

4. The balancing force is equal to the resultant force, but ***opposite*** in direction as shown in Fig. 21.6 (*a*). Since the balancing force is proportional to $m.r$, therefore

$$m \times 0.2 = \text{vector } ea = 23 \text{ kg-m} \quad \text{or} \quad m = 23/0.2 = \textbf{115 kg Ans.}$$

By measurement we also find that the angle of inclination of the balancing mass (m) from the horizontal mass of 200 kg

$$\theta = 201° \quad \textbf{Ans.}$$

21.6. Balancing of Several Masses Rotating in Different Planes

Example 2. *A shaft carries four masses A, B, C and D of magnitude 200 kg, 300 kg, 400 kg and 200 kg respectively and revolving at radii 80 mm, 70 mm, 60 mm and 80 mm in planes measured from A at 300 mm, 400 mm and 700 mm. The angles between the cranks measured anticlockwise are A to B 45°, B to C 70° and C to D 120°. The balancing masses are to be placed in planes X and Y. The distance between the planes A and X is 100 mm, between X and Y is 400 mm and between Y and D is 200 mm. If the balancing masses revolve at a radius of 100 mm, find their magnitudes and angular positions.*

Solution. Given : $m_A = 200$ kg ; $m_B = 300$ kg ; $m_C = 400$ kg ; $m_D = 200$ kg ; $r_A = 80$ mm $= 0.08$m ; $r_B = 70$ mm $= 0.07$ m ; $r_C = 60$ mm $= 0.06$ m ; $r_D = 80$ mm $= 0.08$ m ; $r_X = r_Y = 100$ mm $= 0.1$ m

Let m_X = Balancing mass placed in plane X, and

m_Y = Balancing mass placed in plane Y.

The position of planes and angular position of the masses (assuming the mass A as horizontal) are shown in Fig. 21.8 (*a*) and (*b*) respectively.

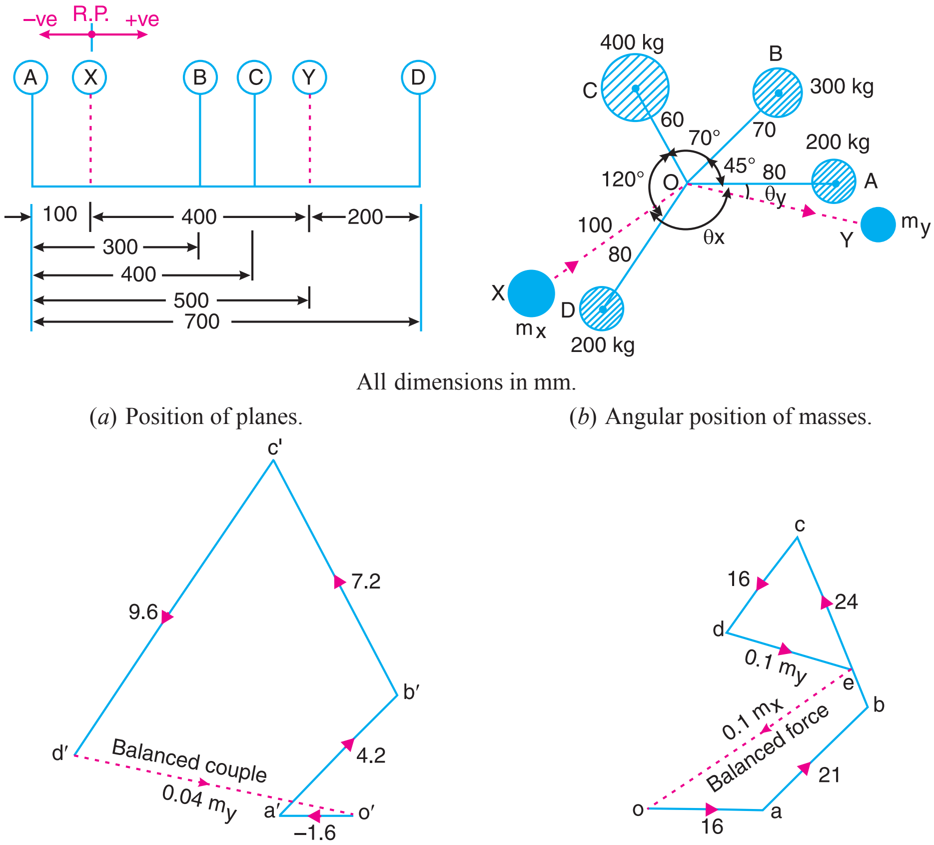
Assume the plane X as the reference plane (*R.P.*). The distances of the planes to the right of plane X are taken as + ve while the distances of the planes to the left of plane X are taken as – ve.

Table 21.2

Plane (1)	Mass (m) kg (2)	Radius (r) m (3)	Cent.force ÷ ω^2 (m.r) kg-m (4)	Distance from Plane x(l) m (5)	Couple ÷ ω^2 (m.r.l) kg-m^2 (6)
A	200	0.08	16	– 0.1	– 1.6
X(*R.P.*)	m_X	0.1	0.1 m_X	0	0
B	300	0.07	21	0.2	4.2
C	400	0.06	24	0.3	7.2
Y	m_Y	0.1	0.1 m_Y	0.4	0.04 m_Y
D	200	0.08	16	0.6	9.6

1. First of all, draw the couple polygon from the data given in Table 21.2 (column 6) as shown in Fig. 21.8 (*c*) to some suitable scale. The vector $d'o'$ represents the balanced couple. Since the balanced couple is proportional to 0.04 m_Y, therefore by measurement,

$$0.04\, m_Y = \text{vector } d'o' = 7.3 \text{ kg-m}^2 \quad \text{or} \quad m_Y = 182.5 \text{ kg}$$ **Ans.**



All dimensions in mm.

(*a*) Position of planes. (*b*) Angular position of masses.

(*c*) Couple polygon. (*d*) Force polygon.

Fig. 21.8

By measurement, the angular position of m_Y is $\theta_Y = 12°$ in the clockwise direction from mass m_A (*i.e.* 200 kg). **Ans.**

2. Now draw the force polygon from the data given in Table 21.2 (column 4) as shown in Fig. 21.8 (*d*). The vector *eo* represents the balanced force. Since the balanced force is proportional to 0.1 m_X, therefore by measurement,

$$0.1\, m_X = \text{vector } eo = 35.5 \text{ kg-m} \quad \text{or} \quad m_X = 355 \text{ kg}$$ **Ans.**

By measurement, the angular position of m_X is $\theta_X = 145°$ in the clockwise direction from mass m_A (*i.e.* 200 kg). **Ans.**

Table 21.3

Example 3. *A, B, C and D are four masses carried by a rotating shaft at radii 100, 125, 200 and 150 mm respectively. The planes in which the masses revolve are spaced 600 mm apart and the mass of B, C and D are 10 kg, 5 kg, and 4 kg respectively.*

Find the required mass A and the relative angular settings of the four masses so that the shaft shall be in complete balance.

Solution. Given : r_A = 100 mm = 0.1 m ; r_B = 125 mm = 0.125 m ; r_C = 200 mm = 0.2 m ; r_D = 150 mm = 0.15 m ; m_B = 10 kg ; m_C = 5 kg ; m_D = 4 kg

The position of planes is shown in Fig. 21.10 (*a*). Assuming the plane of mass *A* as the reference plane (*R.P.*), the data may be tabulated as below :

Table 21.4

Plane (1)	*Mass (m)* kg (2)	*Radius (r)* m (3)	*Cent. Force* $\div \omega^2$ *(m.r)* kg-m (4)	*Distance from plane A (l)* m (5)	*Couple* $\div \omega^2$ *(m.r.l)* kg-m^2 (6)
A(R.P.)	m_A	0.1	0.1 m_A	0	0
B	10	0.125	1.25	0.6	0.75
C	5	0.2	1	1.2	1.2
D	4	0.15	0.6	1.8	1.08

Drawing the couple polygon from the data given in Table 21.4 (column 6). Assume the position of mass *B* in the horizontal direction. By measurement, we find that the angular setting of mass *C* from mass *B* in the anticlockwise direction, *i.e.*

$$\angle BOC = 240° \quad \textbf{Ans.}$$

and angular setting of mass *D* from mass *B* in the anticlockwise direction, *i.e* $\angle BOD = 100°$ **Ans.**

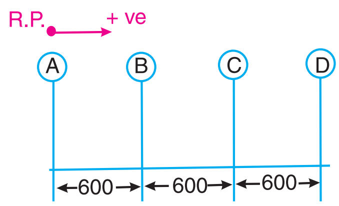
Draw the force polygon to some suitable scale, as shown in Fig. 21.10 (*d*).

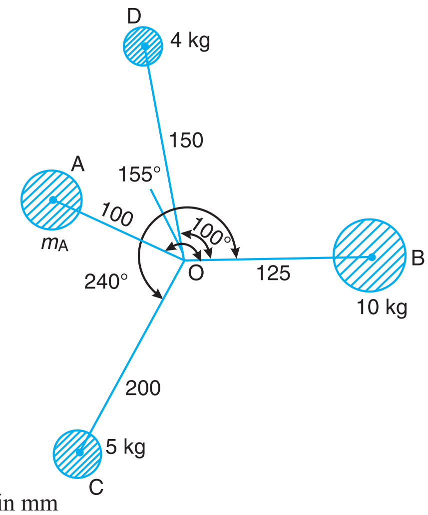
Since the closing side of the force polygon (vector *do*) is proportional to 0.1 m_A, therefore by measurement,

$$0.1\ m_A = 0.7 \text{ kg-m}^2 \quad \text{or} \quad m_A = 7 \text{ kg} \quad \textbf{Ans.}$$

Now draw *OA* in Fig. 21.10 (*b*), parallel to vector *do*. By measurement, we find that the angular setting of mass *A* from mass *B* in the anticlockwise direction, *i.e.*

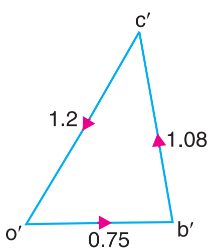
$$\angle BOA = 155° \quad \textbf{Ans.}$$

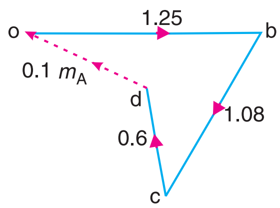




All dimensions in mm

(*a*) Position of planes. (*b*) Angular position of masses.





Explain about primary and secondary unbalanced forces of rotating masses ?

22.1. Introduction

BALANCING OF RECIPROCATING MASSES:

The resultant of all the forces acting on the body of the engine due to inertia forces only is known as ***unbalanced force*** or ***shaking force***.

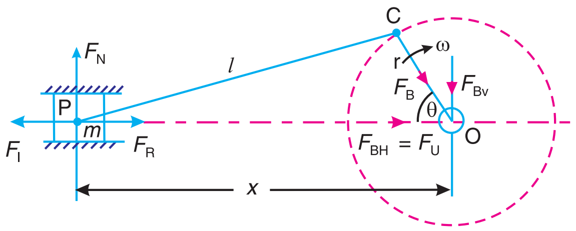


Fig. 22.1. Reciprocating engine mechanism.

Let F_R = Force required to accelerate the reciprocating parts,

F_I = Inertia force due to reciprocating parts,

F_N = Force on the sides of the cylinder walls or normal force acting on the cross-head guides, and

F_B = Force acting on the crankshaft bearing or main bearing.

22.2. Primary and Secondary Unbalanced Forces of Reciprocating Masses

Consider a reciprocating engine mechanism as shown in Fig. 22.1.

Let m = Mass of the reciprocating parts,

l = Length of the connecting rod PC,

r = Radius of the crank OC,

θ = Angle of inclination of the crank with the line of stroke PO,

ω = Angular speed of the crank,

n = Ratio of length of the connecting rod to the crank radius = l / r.

The acceleration of the reciprocating parts is approximately given by the expression,

$$a_R = \omega^2 \cdot r\left(\cos\theta + \frac{\cos 2\theta}{n}\right)$$

∴ Inertia force due to reciprocating parts or force required to accelerate the reciprocating parts,

$$F_I = F_R = \text{Mass} \times \text{acceleration} = m \cdot \omega^2 \cdot r\left(\cos\theta + \frac{\cos 2\theta}{n}\right)$$

The horizontal component of the force exerted on the crank shaft bearing (*i.e.* F_{BH}) is equal and opposite to inertia force (F_I). This force is an unbalanced one and is denoted by F_U.

∴ Unbalanced force,

$$F_U = m \cdot \omega^2 \cdot r\left(\cos\theta + \frac{\cos 2\theta}{n}\right) = m.\omega^2 \cdot r\cos\theta + m \cdot \omega^2 \cdot r \times \frac{\cos 2\theta}{n} = F_P + F_S$$

The expression $(m \cdot \omega^2 \cdot r\cos\theta)$ is known as ***primary unbalanced force*** and $\left(m \cdot \omega^2 \cdot r \times \frac{\cos 2\theta}{n}\right)$ is called ***secondary unbalanced force***.

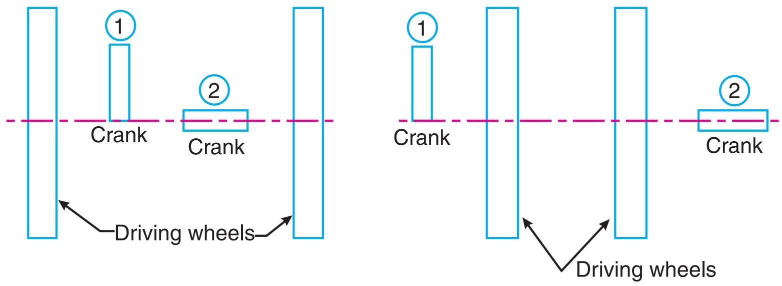
22.4. Partial Balancing of Locomotives

The locomotives, usually, have two cylinders with cranks placed at right angles to each other in order to have uniformity in turning moment diagram. The two cylinder locomotives may be classified as :

1. Inside cylinder locomotives ; and **2.** Outside cylinder locomotives.

In the ***inside cylinder locomotives***, the two cylinders are placed in between the planes of two driving wheels as shown in Fig. 22.3 (*a*) ; whereas in the ***outside cylinder locomotives***, the two cylinders are placed outside the driving wheels, one on each side of the driving wheel, as shown in Fig. 22.3 (*b*). The locomotives may be

(*a*) Single or uncoupled locomotives ; and **(*b*)** Coupled locomotives.



(*a*) Inside cylinder locomotives. (*b*) Outside cylinder locomotives.

Fig. 22.3

22.5. Effect of Partial Balancing of Reciprocating Parts of Two Cylinder Locomotives

22.6. Variation of Tractive Force

The resultant unbalanced force due to the two cylinders, along the line of stroke, is known as ***tractive force***.

$\therefore$ As per definition, the tractive force,

$$F_T = \text{Resultant unbalanced force along the line of stroke}$$

$$= (1-c)m.\omega^2.r\cos\theta + (1-c)m.\omega^2.r\cos(90^\circ+\theta)$$

$$= (1-c)m.\omega^2.r(\cos\theta - \sin\theta)$$

Thus, the tractive force is maximum or minimum when θ = 135° or 315°.

$\therefore$ Maximum and minimum value of the tractive force or the variation in tractive force

$$= \pm(1-c)\,m.\omega^2.r(\cos 135^\circ - \sin 135^\circ) = \pm\sqrt{2}\,(1-c)\,m.\omega^2.r$$

22.7. Swaying Couple

The unbalanced forces along the line of stroke for the two cylinders constitute a couple about the centre line *YY* between the cylinders as shown in Fig. 22.5.

This couple has swaying effect about a vertical axis, and tends to sway the engine alternately in clockwise and anticlockwise directions. Hence the couple is known as ***swaying couple.***

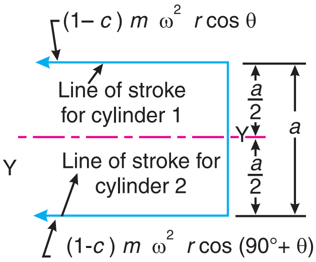
Let a = Distance between the centre lines of the two cylinders.

$\therefore$ Swaying couple

$$= (1-c)m.\omega^2.r\cos\theta \times \frac{a}{2}$$

$$-(1-c)m.\omega^2.r\cos(90°+\theta)\,\frac{a}{2}$$

$$=(1-c)m.\omega^2.r\times\frac{a}{2}(\cos\theta+\sin\theta)$$



Thus, the swaying couple is maximum or minimum when θ = 45° or 225°.

∴ Maximum and minimum value of the swaying couple

$$=\pm(1-c)m.\omega^2.r\times\frac{a}{2}(\cos 45°+\sin 45°)=\pm\frac{a}{\sqrt{2}}(1-c)m.\omega^2.r$$

22.8. Hammer Blow

The maximum magnitude of the unbalanced force along the perpendicular to the line of stroke is known as ***hammer blow***.

∴ Hammer blow = $B.\omega^2.b$

Example 4. *An inside cylinder locomotive has its cylinder centre lines 0.7 m apart and has a stroke of 0.6 m. The rotating masses per cylinder are equivalent to 150 kg at the crank pin, and the reciprocating masses per cylinder to 180 kg. The wheel centre lines are 1.5 m apart. The cranks are at right angles. The whole of the rotating and 2/3 of the reciprocating masses are to be balanced by masses placed at a radius of 0.6 m. Find the magnitude and direction of the balancing masses. Find the fluctuation in rail pressure under one wheel, variation of tractive effort and the magnitude of swaying couple at a crank speed of 300 r.p.m.*

Solution. Given : a = 0.7 m; $l_B = l_C$ = 0.6 m or $r_B = r_C$ = 0.3 m; m_1 = 150 kg; m_2 = 180 kg; c = 2/3; $r_A = r_D$ = 0.6 m; N = 300 r.p.m. or $\omega = 2\pi\times300/60$ = 31.42 rad/s

The equivalent mass of the rotating parts to be balanced per cylinder at the crank pin,

$$m=m_B=m_C=m_1+c.m_2=150+\frac{2}{3}\times180=270\text{ kg}$$

Magnitude and direction of the balancing masses

Let m_A and m_D = Magnitude of the balancing masses

θ_A and θ_D = Angular position of the balancing masses m_A and m_D from the first crank B.

The magnitude and direction of the balancing masses may be determined graphically as discussed below :

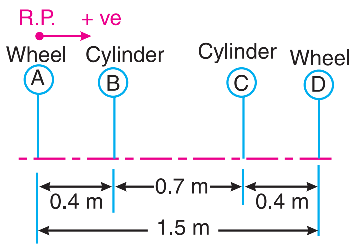
1. First of all, draw the space diagram to show the positions of the planes of the wheels and the cylinders, as shown in Fig. 22.7 (*a*). Since the cranks of the cylinders are at right angles, therefore assuming the position of crank of the cylinder B in the horizontal direction, draw OC and OB at right angles to each other as shown in Fig. 22.7 (*b*).
2. Tabulate the data as given in the following table. Assume the plane of wheel A as the reference plane.

Table 22.1

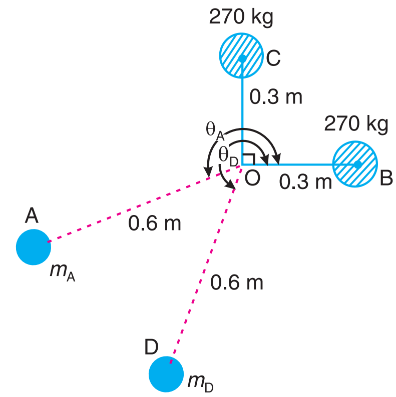
Plane (1)	*mass. (m) kg* (2)	*Radius (r)m* (3)	*Cent. force ÷ ω^2 (m.r) kg-m* (4)	*Distance from plane A (l)m* (5)	*Couple ÷ ω^2 (m.r.l) kg-m^2* (6)
A (*R.P.*)	m_A	0.6	0.6 m_A	0	0
B	270	0.3	81	0.4	32.4
C	270	0.3	81	1.1	89.1
D	m_D	0.6	0.6m_D	1.5	0.9 m_D

3. Now, draw the couple polygon from the data given in Table 22.1 (column 6), to some suitable scale, as shown in Fig 22.7 (*c*). The closing side $c'o'$ represents the balancing couple and it is proportional to 0.9 m_D. Therefore, by measurement,

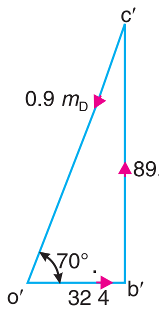
$$0.9\,m_D=\text{vector }c'o'=94.5\text{ kg-m}^2\quad\text{or}\quad m_D=105\text{ kg}\quad\textbf{Ans.}$$



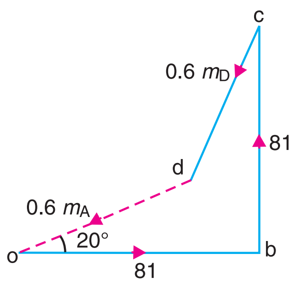
(*a*) Position of planes.



(*b*) Angular position of masses.



(*c*) Couple polygon.



(*d*) Force polygon.

Fig. 22.7

4. To determine the angular position of the balancing mass *D*, draw *OD* in Fig. 22.7 (*b*) parallel to vector $c'o'$. By measurement,

$$\theta_D = 250° \textbf{ Ans.}$$

5. In order to find the balancing mass *A*, draw the force polygon from the data given in Table 22.1 (column 4), to some suitable scale, as shown in Fig. 22.7 (*d*), The vector *do* represents the balancing force and it is proportional to 0.6 m_A. Therefore by measurement,

$$0.6\, m_A = \text{vector } do = 63 \text{ kg-m or } m_A = 105 \text{ kg} \textbf{ Ans.}$$

6. To determine the angular position of the balancing mass *A*, draw *OA* in Fig. 22.7 (*b*) parallel to vector *do*. By measurement,

$$\theta_A = 200° \textbf{ Ans.}$$

Fluctuation in rail pressure

We know that each balancing mass = 105 kg

Balancing mass for rotating masses,

$$D = \frac{m_1}{m} \times 105 = \frac{150}{270} \times 105 = 58.3 \text{ kg}$$

and balancing mass for reciprocating masses,

$$B = \frac{c.m_2}{m} \times 105 = \frac{2}{3} \times \frac{180}{270} \times 105 = 46.6 \text{ kg}$$

This balancing mass of 46.6 kg for reciprocating masses gives rise to the centrifugal force.

∴ Fluctuation in rail pressure or hammer blow

$$= B.\omega^2.b = 46.6\,(31.42)^2\, 0.6 = 27\,602 \text{ N. } \textbf{Ans.} \qquad \ldots (\because\ b = r_A = r_D)$$

Variation of tractive ef fort

We know that maximum variation of tractive ef fort

$$= \pm\sqrt{2}(1-c)m_2.\omega^2.r = \pm\sqrt{2}\left(1-\frac{2}{3}\right)180(31.42)^2 0.3 \text{ N}$$

$= \pm 25\ 127$ N **Ans.** ... ($\because r = r_B = r_C$)

Swaying couple

We know that maximum swaying couple

$$= \frac{a(1-c)}{\sqrt{2}} \times m_2.\omega^2.r = \frac{0.7\left(1-\frac{2}{3}\right)}{\sqrt{2}} \times 180(31.42)^2 0.3 \text{ N-m}$$

$= 8797$ N-m **Ans.**

$$\omega = \frac{v}{D/2} = \frac{33.33}{1.8/2} = 37 \text{rad/s}$$

We know that hammer blow

$$= \pm B.\omega^2.b = 33(37)^2 0.675 = \pm 30.494 \text{ N} \quad \textbf{Ans.}$$

. . . ($\because B = m''_E$, and $b = r_B = r_E$)

22.10. Balancing of Primary Forces of Multi-cylinder In-line Engines

The multi-cylinder engines with the cylinder centre lines in the same plane and on the same side of the centre line of the crankshaft, are known as ***In-line engines.*** The following two conditions must be satisfied in order to give the primary balance of the reciprocating parts of a multi-cylinder engine :

1. The algebraic sum of the primary forces must be equal to zero. In other words, the primary force polygon must *close ; and

2. The algebraic sum of the couples about any point in the plane of the primary forces must be equal to zero. In other words, the primary couple polygon must close.

22.11. Balancing of Secondary Forces of Multi-cylinder In-line Engines

When the connecting rod is not too long (*i.e.* when the obliquity of the connecting rod is considered), then the secondary disturbing force due to the reciprocating mass arises.

The secondary force,

$$F_S = m.\omega^2.r \times \frac{\cos 2\theta}{n}$$

Example 5. *A four cylinder vertical engine has cranks 150 mm long. The planes of rotation of the first, second and fourth cranks are 400 mm, 200 mm and 200 mm respectively from the third crank and their reciprocating masses are 50 kg, 60 kg and 50 kg respectively. Find the mass of the reciprocating parts for the third cylinder and the relative angular positions of the cranks in order that the engine may be in complete primary balance.*

Solution. Given $r_1 = r_2 = r_3 = r_4 = 150$ mm $= 0.15$ m ; $m_1 = 50$ kg ; $m_2 = 60$ kg ; $m_4 = 50$ kg

The position of planes is shown in Fig. 22.17 (*a*). Assuming the plane of third cylinder as the reference plane, the data may be tabulated as given in Table 22.8.

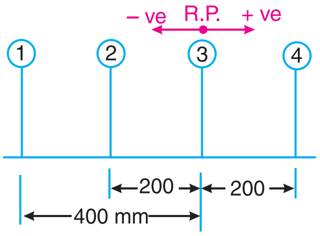
Table 22.8

Plane (1)	*Mass (m) kg* (2)	*Radius (r) m* (3)	*Cent. force* $\div \omega^2$ *(m.r) kg-m* (4)	*Distance from plane 3(l) m* (5)	*Couple* $\div \omega^2$ *(m.r.l) kg-m*^{2} (6)
1	50	0.15	7.5	– 0.4	– 3
2	60	0.15	9	– 0.2	– 1.8
3(*R.P.*)	m_3	0.15	$0.15m_3$	0	0
4	50	0.15	7.5	0.2	1.5

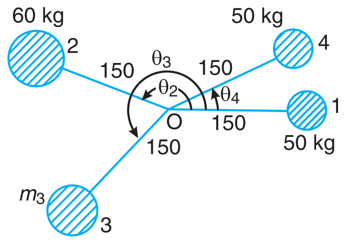
First of all, the angular position of cranks 2 and 4 are obtained by drawing the couple polygon from the data given in Table 22.8 (column 6). Assume the position of crank 1 in the horizontal direction as shown in Fig 22.17 (*b*), The couple polygon, as shown in Fig. 22.17 (*c*), is drawn as discussed below:

1. Draw vector $o'a'$ in the horizontal direction (*i.e.* parallel to $O1$) and equal to – 3 kg-m^2, to some suitable scale.

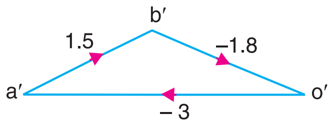
2. From point o' and a', draw vectors $o'b'$ and $a'b'$ equal to – 1.8 kg-m^2 and 1.5 kg-m^2 respectively. These vectors intersect at b'.



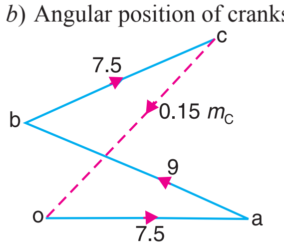
(*a*) Position of planes.



(*b*) Angular position of cranks.



(*c*) Couple polygon.



(*d*) Force polygon.

Fig. 22.17

3. Now in Fig. 22.17 (*b*), draw *O*2 parallel to vector $o'b'$ and *O*4 parallel to vector $a'b'$.

By measurement, we find that the angular position of crank 2 from crank 1 in the anticlockwise direction is

$$\theta_2 = 160° \quad \textbf{Ans.}$$

and the angular position of crank 4 from crank 1 in the anticlockwise direction is

$$\theta_4 = 26° \quad \textbf{Ans.}$$

In order to find the mass of the third cylinder (m_3) and its angular position, draw the force polygon, to some suitable scale, as shown in Fig. 22.17 (*d*), from the data given in Table 22.8 (column 4). Since the closing side of the force polygon (vector *co*) is proportional to 0.15 m_3, therefore by measurement,

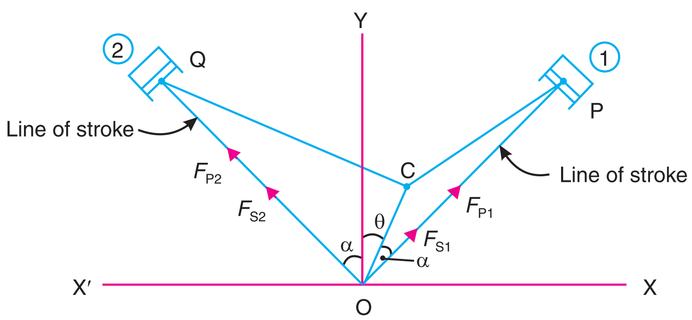
$$0.15 m_3 = 9 \text{ kg-m} \quad \text{or} \quad m_3 = 60 \text{ kg} \quad \textbf{Ans.}$$

Now draw O3 in Fig 22.17 (b), parallel to vector co. By measurement, we find that theangular position of crank 3 from crank 1 in the anticlockwise direction is θ3 = 227° Ans.

22.13. Balancing of V-engines

Consider a symmetrical two cylinder *V*-engine as shown in Fig. 22.33, The common crank *OC* is driven by two connecting rods *PC* and *QC*. The lines of stroke *OP* and *OQ* are inclined to the vertical *OY*, at an angle α as shown in Fig 22.33.

Let
m = Mass of reciprocating parts per cylinder ,
l = Length of connecting rod,
r = Radius of crank,
n = Ratio of length of connecting rod to crank radius = l / r
θ = Inclination of crank to the vertical at any instant,
ω = Angular velocity of crank.



We know that inertia force due to reciprocating parts of cylinder 1, along the line of stroke

$$= m.\omega^2.r\left[\cos(\alpha-\theta)+\frac{\cos 2(\alpha-\theta)}{n}\right]$$

and the inertia force due to reciprocating parts of cylinder 2, along the line of stroke

$$= m.\omega^2.r\left[\cos(\alpha-\theta)+\frac{\cos 2(\alpha+\theta)}{n}\right]$$

The balancing of V-engines is only considered for primary and secondary forces * as discussed below :

Considering primary forces

We know that primary force acting along the line of stroke of cylinder 1,

$$F_{P1} = m.\omega^2.r\cos(\alpha-\theta)$$

$\therefore$ Component of F_{P1} along the vertical line OY,

$$= F_{P1}\cos\alpha = m.\omega^2 r.\cos(\alpha-\theta)\cos\alpha \qquad \ldots (i)$$

and component of F_{P1} along the horizontal line OX

$$= F_{P1}\sin\alpha = m.\omega^2 r\cos(\alpha-\theta)\sin\alpha \qquad \ldots (ii)$$

Similarly, primary force acting along the line of stroke of cylinder 2,

$$F_{P2} = m.\omega^2.r\cos(\alpha+\theta)$$

$\therefore$ Component of F_{P2} along the vertical line OY

$$= F_{P2}\cos\alpha = m.\omega^2.r\cos(\alpha+\theta)\cos\alpha \qquad \ldots (iii)$$

and component of F_{P2} along the horizontal line OX'

$$= F_{P2}\sin\alpha = m.\omega^2.r\cos(\alpha+\theta)\sin\alpha \qquad \ldots (iv)$$

Total component of primary force along the vertical line OY

$$F_{PV} = (i) + (iii) = m.\omega^2.r\cos\alpha\,[\cos(\alpha-\theta)+\cos(\alpha+\theta)]$$

$$= m.\omega^2.r\cos\alpha\times 2\cos\alpha\cos\theta$$

$$\ldots [\because \cos(\alpha-\theta)+\cos(\alpha+\theta) = 2\cos\alpha\cos\theta]$$

$$= 2\,m.\omega^2.r\cos^2\alpha.\cos\theta$$

and total component of primary force along the horizontal line OX

$$F_{PH} = (ii) - (iv) = m.\omega^2.r\sin\alpha[\cos(\alpha-\theta)-\cos(\alpha+\theta)]$$

$$= m.\omega^2.r\sin\alpha\times 2\sin\alpha\sin\theta$$

$$\ldots [\because \cos(\alpha-\theta)-\cos(\alpha+\theta) = 2\sin\alpha\sin\theta]$$

$$= 2m.\omega^2.r\sin^2\alpha.\sin\theta$$

$\therefore$ Resultant primary force,

$$F_P = \sqrt{(F_{PV})^2+(F_{PH})^2}$$

$$= 2m.\omega^2.r\sqrt{(\cos^2\alpha.\cos\theta)^2+(\sin^2\alpha.\sin\theta)^2} \qquad \ldots (v)$$

Notes : The following results, derived from equation **(*v*)**, depending upon the value of α may be noted :

1. When $2\alpha = 60°$ or $\alpha = 30°$,

$$F_P = 2m.\omega^2.r\sqrt{(\cos^2 30°\cos\theta)^2+(\sin^2 30°\sin\theta)^2}$$

$$2m.\omega^2.r\sqrt{\left(\frac{3}{4}\cos\theta\right)^2+\left(\frac{1}{4}\sin\theta\right)^2} = \frac{m}{2}\times\omega^2.r\sqrt{9\cos^2\theta+\sin^2\theta} \quad \text{...}(vi)$$

2. When $2\alpha = 90°$ or $\alpha = 45°$

$$F_P = 2m.\omega^2.r\sqrt{(\cos^2 45°\cos\theta)^2+(\sin^2 45°\sin\theta)^2}$$

$$= 2m.\omega^2.r\sqrt{\left(\frac{1}{2}\cos\theta)\right)^2+\left(\frac{1}{2}\sin\theta\right)^2} = m.\omega^2.r \quad \text{. . . }(vii)$$

3. When $2\alpha = 120°$ or $\alpha = 60°$,

$$F_P = 2m.\omega^2.r\sqrt{(\cos^2 60°\cos\theta)^2+(\sin^2 60°\sin\theta)^2}$$

$$= 2m.\omega^2.r\sqrt{\left(\frac{1}{4}\cos\theta\right)^2+\left(\frac{3}{4}\sin\theta\right)^2} = \frac{m}{2}\times\omega^2.r\sqrt{\cos^2\theta+9\sin^2\theta} \quad \text{... }(viii)$$

Considering secondary forces

We know that secondary force acting along the line of stroke of cylinder 1,

$$F_{S1} = m.\omega^2.r\times\frac{\cos 2(\alpha-\theta)}{n}$$

$\therefore$ Component of F_{S1} along the vertical line OY

$$= F_{S1}\cos\alpha = m.\omega^2.r\times\frac{\cos 2(\alpha-\theta)}{n}\times\cos\alpha \quad \text{. . . }(ix)$$

and component of F_{S1} along the horizontal line OX

$$= F_{S1}\sin\alpha = m.\omega^2.r\times\frac{\cos 2(\alpha-\theta)}{n}\times\sin\alpha \quad \text{. . . }(x)$$

Similarly, secondary force acting along the line of stroke of cylinder 2,

$$F_{S2} = m.\omega^2 r\times\frac{\cos 2(\alpha+\theta)}{n}$$

$\therefore$ Component of F_{S2} along the vertical line OY

$$= F_{S2}\cos\alpha = m.\omega^2.r\times\frac{\cos 2(\alpha+\theta)}{n}\times\cos\alpha \quad \text{. . . }(xi)$$

and component of F_{S2} along the horizontal line OX'

$$= F_{S2}\sin\alpha = m.\omega^2.r\times\frac{\cos 2(\alpha+\theta)}{n}\times\sin\alpha \quad \text{. . . }(xii)$$

Total component of secondary force along the vertical line OY,

$$F_{SV} = (ix) + (xi) = \frac{m}{n}\times\omega^2.r\cos\alpha\,[\cos 2(\alpha-\theta)+\cos 2(\alpha+\theta)]$$

$$= \frac{m}{n}\times\omega^2.r\cos\alpha\times 2\cos 2\alpha\cos 2\theta = \frac{2m}{n}\times\omega^2.r\cos\alpha.\cos 2\alpha\cos 2\theta$$

and total component of secondary force along the horizontal line OX,

$$F_{SH} = (x) - (xii) = \frac{m}{n}\times\omega^2.r\sin\alpha\,[\cos 2(\alpha-\theta)-\cos 2(\alpha+\theta)]$$

$$= \frac{m}{n}\times\omega^2.r\sin\alpha\times 2\sin 2\alpha.\sin 2\theta$$

$$= \frac{2m}{n}\times\omega^2.r\sin\alpha.\sin 2\alpha.\sin 2\theta$$

$\therefore$ Resultant secondary force,

$$F_S = \sqrt{(F_{SV})^2+(F_{SH})^2}$$

$$= \frac{2m}{n}\times\omega^2.r\sqrt{(\cos\alpha.\cos 2\alpha.\cos 2\theta)^2+(\sin\alpha.\sin 2\alpha.\sin 2\theta)^2}$$

$$\text{. . . }(xiii)$$

UNIT-V

MECHANICAL VIBRATIONS

When elastic bodies such as a spring, a beam and ashaft are displaced from the equilibrium position by the application of external forces, and then released, they execute a vibratory motion.

1. ***Period of vibration or time period*** **.** It is the time interval after which the motion is repeated itself.
2. ***Cycle.*** It is the motion completed during one time period.
3. ***Frequency.*** It is the number of cycles described in one second. In S.I. units, the frequency is expressed in hertz (Hz)

23.3. Types of Vibratory Motion

The following types of vibratory motion are important from the subject point of view :

1. ***Free or natural vibrations.*** When no external force acts on the body , after giving it an initial displacement, then the body is said to be under ***free or natural vibrations.***

2. ***Forced vibrations.*** When the body vibrates under the influence of external force, then the body is said to be under ***forced vibrations.***

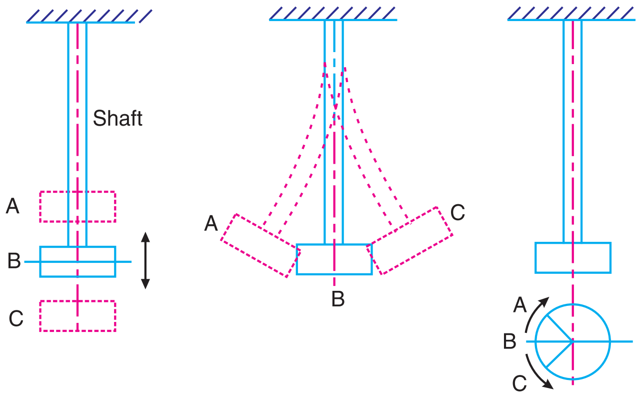
3. ***Damped vibrations.*** When there is a reduction in amplitude over every cycle of vibration, the motion is said to be ***damped vibration.***

23.4. Types of Free Vibrations

The following three types of free vibrations are important from the subject point of view :

1. Longitudinal vibrations, **2.** Transverse vibrations, and **3.** Torsional vibrations.

Consider a weightless constraint (spring or shaft) whose one end is fixed and the other end carrying a heavy disc, as shown in Fig. 23.1.



B = Mean position ; *A* and *C* = Extreme positions.

(*a*) Longitudinal vibrations. (*b*) Transverse vibrations. (*c*) Torsional vibrations.

Fig. 23.1. Types of free vibrations.

1. ***Longitudinal vibrations.*** When the particles of the shaft or disc moves parallel to the axis of the shaft, as shown in Fig. 23.1 (*a*), then the vibrations are known as ***longitudinal vibrations.***

2. ***Transverse vibrations.*** When the particles of the shaft or disc move approximately perpendicular to the axis of the shaft, as shown in Fig. 23.1 (*b*), then the vibrations are known as ***transverse vibrations.*** In this case, the shaft is straight and bent alternately and bending stresses are induced in the shaft.

3. ***Torsional vibrations****. When the particles of the shaft or disc move in a circle about the axis of the shaft, as shown in Fig. 23.1 (*c*), then the vibrations are known as ***torsional vibrations.***

23.5. Natural Frequency of Free Longitudinal Vibrations

23.5. Natural Frequency of Free Longitudinal Vibrations

The natural frequency of the free longitudinal vibrations may be determined by the following three methods :

1. *Equilibrium Method*

Consider a constraint (*i.e.* spring) of negligible mass in an unstrained position, as shown in Fig. 23.2 (*a*).

Let s = Stiffness of the constraint. It is the force required to produce unit displacement in the direction of vibration. It is usually expressed in N/m.

m = Mass of the body suspended from the constraint in kg,

W = Weight of the body in newtons = $m.g$,

δ = Static deflection of the spring in metres due to weight W newtons, and

x = Displacement given to the body by the external force, in metres.

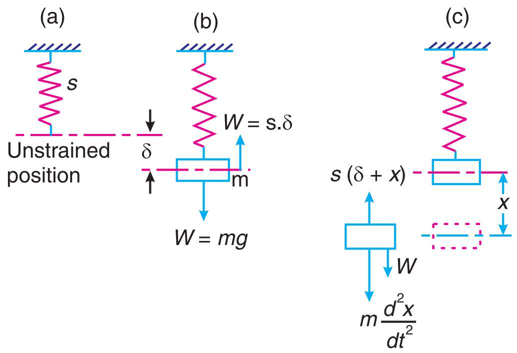


Fig. 23.2. Natural frequency of free longitudinal vibrations.

In the equilibrium position, as shown in Fig. 23.2 (*b*), the gravitational pull $W = m.g$, is balanced by a force of spring, such that $W = s.\delta$.

Since the mass is now displaced from its equilibrium position by a distance x, as shown in Fig. 23.2 (*c*), and is then released, therefore after time t,

Restoring force $= W - s(\delta + x) = W - s.\delta - s.x$

$= s.\delta - s.\delta - s.x = -s.x$ . . . $(\because W = s.\delta)$ . . . (*i*)

. . . (Taking upward force as negative)

and Accelerating force = Mass × Acceleration

$$= m \times \frac{d^2x}{dt^2}$$ . . . (Taking downward force as positive) . . . (*ii*)

Equating equations (*i*) and (*ii*), the equation of motion of the body of mass m after time t is

$$m \times \frac{d^2x}{dt^2} = -s.x \quad \text{or} \quad m \times \frac{d^2x}{dt^2} + s.x = 0$$

$$\therefore \quad \frac{d^2x}{dt^2} + \frac{s}{m} \times x = 0 \qquad \ldots (iii)$$

We know that the fundamental equation of simple harmonic motion is

$$\frac{d^2x}{dt^2} + \omega^2.x = 0 \qquad \ldots (iv)$$

Comparing equations (*iii*) and (*iv*), we have

$$\omega = \sqrt{\frac{s}{m}}$$

$$\therefore \quad \text{Time period,} \quad t_p = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{s}}$$

and natural frequency, $f_n = \frac{1}{t_p} = \frac{1}{2\pi}\sqrt{\frac{s}{m}} = \frac{1}{2\pi}\sqrt{\frac{g}{\delta}}$ . . . ($\because\ m.g = s.\delta$)

Taking the value of g as 9.81 m/s^2 and δ in metres,

$$f_n = \frac{1}{2\pi}\sqrt{\frac{9.81}{\delta}} = \frac{0.4985}{\sqrt{\delta}}\text{Hz}$$

Note : The value of static deflection δ may be found out from the given conditions of the problem. For longitudinal vibrations, it may be obtained by the relation,

$$\delta = \frac{W.l}{E.A}$$

where δ = Static deflection *i.e.* extension or compression of the constraint,

W = Load attached to the free end of constraint,

l = Length of the constraint,

E = Young's modulus for the constraint, and

A = Cross-sectional area of the constraint.

2. *Energy method*

In the case of vibrations, the datum position is the mean or equilibrium position at which the potential energy of the body or the system is zero. In the free vibrations, no energy is transferred to the system or from the system. Therefore the summation of kinetic energy and potential energy must be a constant quantity which is same at all the times.

In other words,

$$\therefore \quad \frac{d}{dt}(K.E. + P.E.) = 0$$

We know that kinetic energy,

$$K.E. = \frac{1}{2}\times m\left(\frac{dx}{dt}\right)^2$$

and potential energy,

$$P.E. = \left(\frac{0+s.x}{2}\right)x = \frac{1}{2}\times s.x^2$$

. . . ($\because$ $P.E.$ = Mean force × Displacement)

$$\therefore \quad \frac{d}{dt}\left[\frac{1}{2}\times m\left(\frac{dx}{dt}\right)^2 + \frac{1}{2}\times s.x^2\right] = 0$$

$$\frac{1}{2}\times m\times 2\times\frac{dx}{dt}\times\frac{d^2x}{dt^2} + \frac{1}{2}\times s\times 2x\times\frac{dx}{dt} = 0$$

or $\quad m\times\frac{d^2x}{dt^2} + s.x = 0 \quad$ or $\quad \frac{d^2x}{dt^2} + \frac{s}{m}\times x = 0$ . . . (Same as before)

The time period and the natural frequency may be obtained as discussed in the previous method.

3. *Rayleigh's method*

In this method, the maximum kinetic energy at the mean position is equal to the maximum potential energy (or strain energy) at the extreme position. Assuming the motion executed by the vibration to be simple harmonic, then

$$x = X\sin\omega.t \qquad \text{. . . }(i)$$

where x = Displacement of the body from the mean position after time t seconds, and

X = Maximum displacement from mean position to extreme position.

Now, differentiating equation ***(i)***, we have

$$\frac{dx}{dt} = \omega \times X \cos \omega.t$$

Since at the mean position, $t = 0$, therefore maximum velocity at the mean position,

$$v = \frac{dx}{dt} = \omega.X$$

$\therefore$ Maximum kinetic energy at mean position

$$= \frac{1}{2} \times m.v^2 = \frac{1}{2} \times m.\omega^2.X^2 \qquad \ldots (ii)$$

and maximum potential energy at the extreme position

$$= \left(\frac{0 + s.X}{2}\right) X = \frac{1}{2} \times s.X^2 \qquad \ldots (iii)$$

Equating equations (*ii*) and (*iii*),

$$\frac{1}{2} \times m.\omega^2.X^2 = \frac{1}{2} \times s.X^2 \quad \text{or} \quad \omega^2 = \frac{s}{m}, \text{ and } \omega = \sqrt{\frac{s}{m}}$$

$\therefore$ Time period,

$$t_p = \frac{2\pi}{\omega} = 2\pi \sqrt{\frac{s}{m}} \qquad \ldots \text{(Same as before)}$$

and natural frequency,

$$f_n = \frac{1}{t_p} = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{s}{m}} \qquad \ldots \text{(Same as before)}$$

23.6. Natural Frequency of Free Transverse Vibrations

Consider a shaft of negligible mass, whose one end is fixed and the other end carries a body of weight W, as shown in Fig. 23.3.

Let s = Stiffness of shaft,

δ = Static deflection due to weight of the body,

x = Displacement of body from mean position after time t.

m = Mass of body = W/g

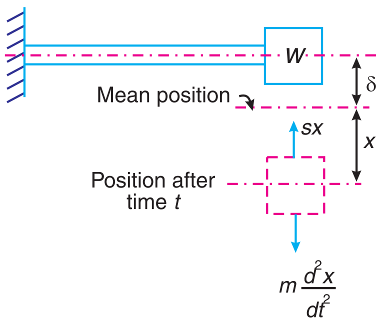


Fig. 23.3. Natural frequency of free transverse vibrations.

As discussed in the previous article,

Restoring force $= - s.x$ $\qquad \ldots (i)$

and accelerating force $= m \times \dfrac{d^2x}{dt^2}$ $\qquad \ldots (ii)$

Equating equations (*i*) and (*ii*), the equation of motion becomes

$$m \times \frac{d^2x}{dt^2} = -s.x \quad \text{or} \quad m \times \frac{d^2x}{dt^2} + s.x = 0$$

$$\therefore \quad \frac{d^2x}{dt^2} + \frac{s}{m} \times x = 0 \qquad \ldots \text{(Same as before)}$$

Hence, the time period and the natural frequency of the transverse vibrations are same as that of longitudinal vibrations. Therefore

Time period,

$$t_p = 2\pi \sqrt{\frac{m}{s}}$$

and natural frequency,

$$f_n = \frac{1}{t_p} = \frac{1}{2\pi}\sqrt{\frac{s}{m}} = \frac{1}{2\pi}\sqrt{\frac{g}{\delta}}$$

Note : The shape of the curve, into which the vibrating shaft deflects, is identical with the static deflection curve of a cantilever beam loaded at the end. The static deflection of a cantilever beam loaded at the free end is

$$\delta = \frac{Wl^3}{3EI}$$

Example 1. *A cantilever shaft 50 mm diameter and 300 mm long has a disc of mass 100 kg at its free end. The Young's modulus for the shaft material is 200 GN/m². Determine the frequency of longitudinal and transverse vibrations of the shaft.*

Solution. Given : d = 50 mm = 0.05 m ; l = 300 mm = 0.03 m ; m = 100 kg ; E = 200 GN/m^2 = 200 × 10^9 N/m^2

We know that cross-sectional area of the shaft,

$$A = \frac{\pi}{4} \times d^2 = \frac{\pi}{4}(0.05)^2 = 1.96 \times 10^{-3}\ \text{m}^2$$

and moment of inertia of the shaft,

$$I = \frac{\pi}{64} \times d^4 = \frac{\pi}{64}(0.05)^4 = 0.3 \times 10^{-6}\ \text{m}^4$$

Frequency of longitudinal vibration

We know that static deflection of the shaft,

$$\delta = \frac{W.l}{A.E} = \frac{100 \times 9.81 \times 0.3}{1.96 \times 10^{-3} \times 200 \times 10^9} = 0.751 \times 10^{-6}\ \text{m}$$

$$\ldots(\because\ W = m.g)$$

∴ Frequency of longitudinal vibration,

$$f_n = \frac{0.4985}{\sqrt{\delta}} = \frac{0.4985}{\sqrt{0.751 \times 10^{-6}}} = 575\ \text{Hz} \quad \textbf{Ans.}$$

Frequency of transverse vibration

We know that static deflection of the shaft,

$$\delta = \frac{W.l^3}{3E.I} = \frac{100 \times 9.81 \times (0.3)^3}{3 \times 200 \times 10^9 \times 0.3 \times 10^{-6}} = 0.147 \times 10^{-3}\ \text{m}$$

∴ Frequency of transverse vibration,

$$f_n = \frac{0.4985}{\sqrt{\delta}} = \frac{0.4985}{\sqrt{0.147 \times 10^{-3}}} = 41\ \text{Hz} \quad \textbf{Ans.}$$

23.7. Effect of Inertia of the Constraint in Longitudinal and Transverse Vibrations

In deriving the expressions for natural frequency of longitudinal and transverse vibrations, we have neglected the inertia of the constraint *i.e.* shaft. We shall now discuss the effect of the inertia of the constraint, as below :

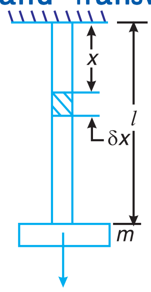


Fig. 23.4. Effect of inertia of the constraint in longitudinal vibrations.

1. *Longitudinal vibration*

Consider the constraint whose one end is fixed and other end is free as shown in Fig. 23.4.

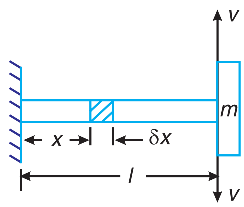
Let l = Length of the constraint,

m_C = Total mass of the constraint

When the mass of the constraint m_C and the mass of the disc m at the end is given, then natural frequency of vibration,

$$f_n = \frac{1}{2\pi}\sqrt{\frac{s}{m + \frac{m_C}{3}}}$$

2. *Transverse vibration*



Consider a constraint whose one end is fixed and the other end is free as shown in Fig. 23.5.

$$f_n = \frac{1}{2\pi}\sqrt{\frac{s}{m + \frac{33\,m_C}{140}}}$$

23.13. Frequency of Free Damped Vibrations (Viscous Damping)

In vibrating systems, the effect of friction is referred to as damping. The damping provided by fluid resistance is known as ***viscous damping.***

The resistance to the motion of the body is provided partly by the medium in which the vibration takes place and partly by the internal friction, and in some cases partly by a dash pot or other external damping device.

Consider a vibrating system, as shown in Fig. 23.17, in which a mass is suspended from one end of the spiral spring and the other end of which is fixed. A damper is provided between the mass and the rigid support.

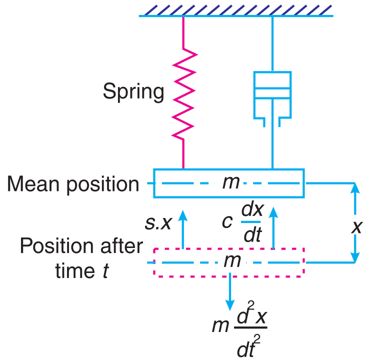


Fig. 23.17. Frequency of free damped vibrations.

Let m = Mass suspended from the spring,

s = Stiffness of the spring,

x = Displacement of the mass from the mean position at time t,

δ = Static deflection of the spring = $m.g/s$, and

c = Damping coefficient or the damping force per unit velocity .

Since in viscous damping, it is assumed that the frictional resistance to the motion of the body is directly proportional to the speed of the movement, therefore

Damping force or frictional force on the mass acting in ***opposite*** direction to the motion of the mass

$$= c \times \frac{dx}{dt}$$

Accelerating force on the mass, acting ***along*** the motion of the mass

$$= m \times \frac{d^2x}{dt^2}$$

and spring force on the mass, acting in ***opposite*** direction to the motion of the mass, $= s.x$

Therefore the equation of motion becomes

$$m \times \frac{d^2x}{dt^2} = -\left(c \times \frac{dx}{dt} + s.x\right)$$

...(Negative sign indicates that the force opposes the motion)

or
$$\frac{d^2x}{dt^2} + \frac{c}{m} \times \frac{dx}{dt} + \frac{s}{m} \times x = 0$$

This is a dif ferential equation of the second order . Assuming a solution of the form $x = e^{k\,t}$ where k is a constant to be determined.

∴ The two roots of the equation are

$$k_1 = -\frac{c}{2m} + \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{s}{m}}$$

and
$$k_2 = -\frac{c}{2m} - \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{s}{m}}$$

The most general solution of the dif ferential equation with its right hand side equal to zero has only complementary function and it is given by

$$x = C_1 e^{k_1 t} + C_2\, e^{k_2 t} \qquad \ldots (ii)$$

where C_1 and C_2 are two arbitrary constants which are to be determined from the initial conditions of the motion of the mass.

1. *When the roots are real (overdamping)*

If $\left(\frac{c}{2m}\right)^2 > \frac{s}{m}$, then the roots k_1 and k_2 are real but negative. This is a case of ***overdamping*** or ***large damping*** and the mass moves slowly to the equilibrium position. This motion is known as ***aperiodic.*** When the roots are real, the most general solution of the dif ferential equation is

$$x = C_1 e^{k_1 t} + C_2 e^{k_2 t}$$

$$= C_1\, e^{\left[-\frac{c}{2m} + \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{s}{m}}\right] t} + C_2\, e^{\left[-\frac{c}{2m} - \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{s}{m}}\right] t}$$

2. *When the roots are complex conjugate (underdamping)*

If $\frac{s}{m} > \left(\frac{c}{2m}\right)^2$, then the radical (*i.e.* the term under the square root) becomes negative. The two roots k_1 and k_2 are then known as complex conjugate. This is a most practical case of damping and it is known as ***underdamping*** or ***small damping.*** The two roots are

$$k_1 = -\frac{c}{2m} + i\sqrt{\frac{s}{m} - \left(\frac{c}{2m}\right)^2}$$

and

$$k_2 = -\frac{c}{2m} - i\sqrt{\frac{s}{m} - \left(\frac{c}{2m}\right)^2}$$

$$x = Ae^{-at} \cos \omega_d .t \qquad \ldots (vi)$$

where

$$\omega_d = \sqrt{\frac{s}{m} - \left(\frac{c}{2m}\right)^2} = \sqrt{(\omega_n)^2 - a^2} \quad ;\text{ and } \quad a = \frac{c}{2m}$$

We see from equation ***(vi),*** that the motion of the mass is simple harmonic whose circular damped frequency is ω_d and the amplitude is Ae^{-at} which diminishes exponentially with time as shown in Fig. 23.18. Though the mass eventually returns to its equilibrium position because of its inertia, yet it overshoots and the oscillations may take some considerable time to die away .

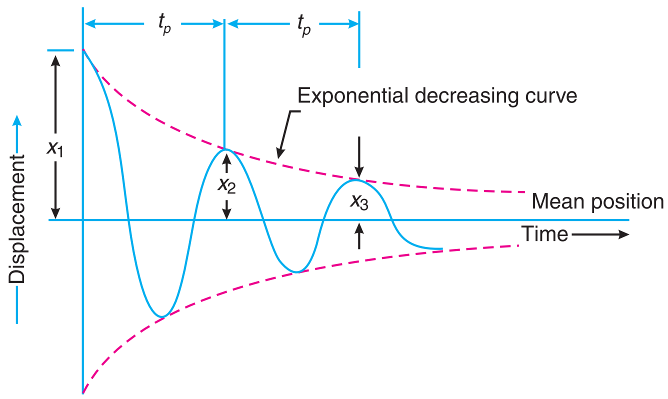


Fig. 23.18. Underdamping or small damping.

We know that the periodic time of vibration,

$$t_p = \frac{2\pi}{\omega_d} = \frac{2\pi}{\sqrt{\frac{s}{m} - \left(\frac{c}{2m}\right)^2}} = \frac{2\pi}{\sqrt{(\omega_n)^2 - a^2}}$$

and frequency of damped vibration,

$$f_d = \frac{1}{t_p} = \frac{\omega_d}{2\pi} = \frac{1}{2\pi}\sqrt{(\omega_n)^2 - a^2} = \frac{1}{2\pi}\sqrt{\frac{s}{m} - \left(\frac{c}{2m}\right)^2} \qquad \ldots (vii)$$

3. *When the roots are equal (critical damping)*

If $\left(\frac{c}{2m}\right)^2 = \frac{s}{m}$, then the radical becomes zero and the two roots k_1 and k_2 are equal. This is a case of **critical damping.** In other words, the critical damping is said to occur when frequency of damped vibration (f_d) is zero (*i.e.* motion is aperiodic). This type of damping is also avoided because the mass moves back rapidly to its equilibrium position, in the shortest possible time.

For critical damping, equation ***(ii)*** may be written as

$$x = (C_1 + C_2)\, e^{-\frac{c}{2m}t} = (C_1 + C_2)\, e^{-\omega_n t} \qquad \ldots \left[\because \frac{c}{2m} = \sqrt{\frac{s}{m}} = \omega_n\right]$$

Thus the motion is again aperiodic. The critical damping coefficient (c_c) may be obtained by substituting c_c for c in the condition for critical damping, *i.e.*

$$\left(\frac{c_c}{2m}\right)^2 = \frac{s}{m} \qquad \text{or} \qquad c_c = 2m\sqrt{\frac{s}{m}} = 2m \times \omega_n$$

The critical damping coefficient is the amount of damping required for a system to be critically damped.

23.14. Damping Factor or Damping Ratio

The ratio of the actual damping coefficient (c) to the critical damping coefficient (c_c) is known as ***damping factor*** **or** ***damping ratio.*** Mathematically,

$$\text{Damping factor} = \frac{c}{c_c} = \frac{c}{2m.\omega_n} \qquad \ldots (\because c_c = 2\pi.\omega_n)$$

23.15. Logarithmic Decrement

It is defined as the natural logarithm of the amplitude reduction factor . The amplitude reduction factor is the ratio of any two successive amplitudes on the same side of the mean position.

If x_1 and x_2 are successive values of the amplitude on the same side of the mean position, as shown in Fig. 23.18, then amplitude reduction factor ,

$$\frac{x_1}{x_2} = \frac{Ae^{-at}}{Ae^{-a(t+t_p)}} = e^{at_p} = \text{constant}$$

where t_p is the period of forced oscillation or the time difference between two consecutive amplitudes. As per definition, logarithmic decrement,

$$\delta = \log\left(\frac{x_1}{x_2}\right) = \log e^{at_p}$$

or

$$\delta = \log_e\left(\frac{x_1}{x_2}\right) = a.t_p = a \times \frac{2\pi}{\omega_d} = \frac{a \times 2\pi}{\sqrt{(\omega_n)^2 - a^2}}$$

$$= \frac{2\pi \times c}{\sqrt{(c_c)^2 - c^2}} \qquad \ldots$$

In general, amplitude reduction factor,

$$\frac{x_1}{x_2} = \frac{x_2}{x_3} = \frac{x_3}{x_4} = \ldots . = \frac{x_n}{x_{n+1}} = e^{at_p} = \text{constant}$$

$\therefore$ Logarithmic decrement,

$$\delta = \log_e\left(\frac{x_n}{x_{n+1}}\right) = a.t_p = \frac{2\pi \times c}{\sqrt{(c_c)^2 - c^2}}$$

Example 2. *The following data are given for a vibratory system with viscous damping:*

Mass = 2.5 kg ; spring constant = 3 N/mm and the amplitude decreases to 0.25 of the initial value after five consecutive cycles.

Determine the damping coefficient of the damper in the system.

Solution. Given : $m = 2.5$ kg ; $s = 3$ N/mm $= 3000$ N/m ; $x_6 = 0.25\ x_1$

We know that natural circular frequency of vibration,

$$\omega_n = \sqrt{\frac{s}{m}} = \sqrt{\frac{3000}{2.5}} = 34.64 \text{ rad/s}$$

Let c = Damping coefficient of the damper in N/m/s,

x_1 = Initial amplitude, and

x_6 = Final amplitude after five consecutive cycles = 0.25 x_1 ...(Given)

We know that

$$\frac{x_1}{x_2} = \frac{x_2}{x_3} = \frac{x_3}{x_4} = \frac{x_4}{x_5} = \frac{x_5}{x_6}$$

or
$$\frac{x_1}{x_6} = \frac{x_1}{x_2} \times \frac{x_2}{x_3} \times \frac{x_3}{x_4} \times \frac{x_4}{x_5} \times \frac{x_5}{x_6} = \left(\frac{x_1}{x_2}\right)^5$$

$\therefore$
$$\frac{x_1}{x_2} = \left(\frac{x_1}{x_6}\right)^{1/5} = \left(\frac{x_1}{0.25\,x_1}\right)^{1/5} = (4)^{1/5} = 1.32$$

We know that

$$\log_e\left(\frac{x_1}{x_2}\right) = a \times \frac{2\pi}{\sqrt{(\omega_n)^2 - a^2}}$$

$$\log_e(1.32) = a \times \frac{2\pi}{\sqrt{(34.64)^2 - a^2}} \quad \text{or} \quad 0.2776 = \frac{a \times 2\pi}{\sqrt{1200 - a^2}}$$

Squaring both sides,

$$0.077 = \frac{39.5\,a^2}{1200 - a^2} \quad \text{or} \quad 92.4 - 0.077\,a^2 = 39.5\,a^2$$

$\therefore$ $a^2 = 2.335$ or $a = 1.53$

We know that $a = c / 2m$ or $c = a \times 2m = 1.53 \times 2 \times 2.5 = 7.65$ N/m/s **Ans.**

Example 3. *The measurements on a mechanical vibrating system show that it has a mass of 8 kg and that the springs can be combined to give an equivalent spring of stiffness 5.4 N/mm. If the vibrating system have a dashpot attached which exerts a force of 40 N when the mass has a velocity of 1 m/s, find :* **1.** *critical damping coefficient,* **2.** *damping factor,* **3.** *logarithmic decrement, and* **4.** *ratio of two consecutive amplitudes.*

Solution. Given : $m = 8$ kg ; $s = 5.4$ N/mm $= 5400$ N/m

Since the force exerted by dashpot is 40 N, and the mass has a velocity of 1 m/s , therefore Damping coefficient (actual),

- $c = 40$ N/m/s

1. *Critical damping coefficient*

We know that critical damping coefficient,

$$c_c = 2m.\omega_n = 2m \times \sqrt{\frac{s}{m}} = 2 \times 8\sqrt{\frac{5400}{8}} = 416 \text{ N/m/s} \quad \textbf{Ans.}$$

2. *Damping factor*

We know that damping factor $= \dfrac{c}{c_c} = \dfrac{40}{416} = 0.096$ **Ans.**

3. *Logarithmic decrement*

We know that logarithmic decrement,

$$\delta = \frac{2\pi c}{\sqrt{(c_c)^2 - c^2}} = \frac{2\pi \times 40}{\sqrt{(416)^2 - (40)^2}} = 0.6 \ \textbf{Ans.}$$

4. *Ratio of two consecutive amplitudes*

Let x_n and x_{n+1} = Magnitude of two consecutive amplitudes,

We know that logarithmic decrement,

$$\delta = \log_e \left[\frac{x_n}{x_{n+1}}\right] \text{ or } \frac{x_n}{x_{n+1}} = e^{\delta} = (2.7)^{0.6} = 1.82 \ \textbf{Ans}$$

Example 4. *The mass of a single degree damped vibrating system is 7.5 kg and makes 24 free oscillations in 14 seconds when disturbed fr om its equilibrium position. The amplitude of vibration reduces to 0.25 of its initial value after five oscillations. Determine :* ***1.*** *stiffness of the spring,* ***2.*** *logarithmic decrement, and* ***3.*** *damping factor, i.e. the ratio of the system damping to critical damping.*

Solution. Given : m = 7.5 kg

Since 24 oscillations are made in 14 seconds, therefore frequency of free vibrations,

$$f_n = 24/14 = 1.7$$

and $$\omega_n = 2\pi \times f_n = 2\pi \times 1.7 = 10.7 \text{ rad/s}$$

1. *Stiffness of the spring*

Let s = Stiffness of the spring in N/m.

We know that $(\omega_n)^2 = s/m$ or $s = (\omega_n)^2 m = (10.7)^2 7.5 = 860$ N/m **Ans.**

2. *Logarithmic decrement*

Let x_1 = Initial amplitude,

x_6 = Final amplitude after five oscillations = 0.25 x_1 ... (Given)

$$\therefore \quad \frac{x_1}{x_6} = \frac{x_1}{x_2} \times \frac{x_2}{x_3} \times \frac{x_3}{x_4} \times \frac{x_4}{x_5} \times \frac{x_5}{x_6} = \left(\frac{x_1}{x_2}\right)^5 \quad \ldots\left[\because \frac{x_1}{x_2} = \frac{x_2}{x_3} = \frac{x_3}{x_4} = \frac{x_4}{x_5} = \frac{x_5}{x_6}\right]$$

$$\frac{x_1}{x_2} = \left(\frac{x_1}{x_6}\right)^{1/5} = \left(\frac{x_1}{0.25\, x_1}\right)^{1/5} = (4)^{1/5} = 1.32$$

We know that logarithmic decrement,

$$\delta = \log_e \left(\frac{x_1}{x_2}\right) = \log_e 1.32 = 0.28 \ \textbf{Ans.}$$

3. *Damping factor*

Let c = Damping coefficient for the actual system, and

c_c = Damping coefficient for the critical damped system.

We know that logarithmic decrement (δ),

$$0.28 = \frac{a \times 2\pi}{\sqrt{(\omega_n)^2 - a^2}} = \frac{a \times 2\pi}{\sqrt{(10.7)^2 - a^2}}$$

$$0.0784 = \frac{a^2 \times 39.5}{114.5 - a^2} \qquad \text{. . . (Squaring both sides)}$$

$8.977 - 0.0784 \quad a^2 = 39.5\ a^2$ or $a^2 = 0.227$ or $a = 0.476$

We know that $a = c / 2m$ or $c = a \times 2m = 0.476 \times 2 \times 7.5 = 7.2$ N/m/s **Ans.**

and $c_c = 2m.\omega_n = 2 \times 7.5 \times 10.7 = 160.5$ N/m/s **Ans.**

$\therefore$ Damping factor = $c/c_c = 7.2 / 160.5 = 0.045$ **Ans.**

23.16. Frequency of Under Damped Forced Vibrations

Consider a system consisting of spring, mass and damper as shown in Fig. 23.19. Let the system is acted upon by an external periodic (*i.e.* simple harmonic) disturbing force,

$$F_x = F \cos \omega.t$$

where F = Static force, and

ω = Angular velocity of the periodic disturbing force.

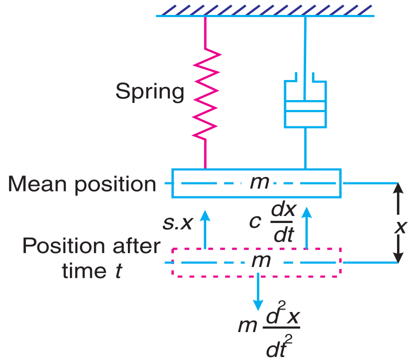


Fig. 23.19. Frequency of under damped forced vibrations.

When the system is constrained to move in vertical guides, it has only one degree of freedom. Let at sometime t, the mass is displaced downwards through a distance x from its mean position.

Using the symbols as discussed in the previous article, the equation of motion may be written as

$$m \times \frac{d^2x}{dt^2} = -c \times \frac{dx}{dt} - s.x + F \cos \omega.t$$

or

$$m \times \frac{d^2x}{dt^2} + c \times \frac{dx}{dt} + s.x = F \cos \omega.t \qquad \text{. . . } (i)$$

This equation of motion may be solved either by differential equation method or by graphical method as discussed below :

1. *Differential equation method*

The equation *(i)* is a differential equation of the second degree whose right hand side is some function in t. The solution of such type of dif ferential equation consists of two parts ; one part is the complementary function and the second is particular integral. Therefore the solution may be written as

$$x = x_1 + x_2$$

where x_1 = Complementary function, and

x_2 = Particular integral.

$\therefore$ The complete solution of the dif ferential equation *(i)* becomes

$$x = x_1 + x_2$$

$$= C.e^{-at} \cos(\omega_d.t - \theta) + \frac{F}{\sqrt{c^2.\omega^2 + (s - m.\omega^2)^2}} \times \cos(\omega.t - \phi)$$

In actual practice, the value of the complementary function x_1 at any time t is much smaller as compared to particular integral x_2. Therefore, the displacement x, at any time t, is given by the particular integral x_2 only.

$$\therefore \quad x = \frac{F}{\sqrt{c^2.\omega^2 + (s - m.\omega^2)^2}} \times \cos(\omega.t - \phi) \qquad \text{... } (vii)$$

$\therefore$ Maximum displacement or the amplitude of forced vibration,

$$x_{max} = \frac{F}{\sqrt{c^2.\omega^2 + (s - m.\omega^2)^2}} \qquad \text{. . . } (viii)$$

Notes : 1. The equations *(vii)* and *(viii)* hold good when steady vibrations of constant amplitude takesplace

23.17. Magnification Factor or Dynamic Magnifier

It is the ratio of ***maximum displacement of the forced vibration (x_{max}) to the deflection due to the static force $F(x_o)$.*** We have proved in the previous article that the maximum displacement or the amplitude of forced vibration,

$$x_{max} = \frac{x_o}{\sqrt{\frac{c^2.\omega^2}{s^2} + \left(1 - \frac{\omega^2}{(\omega_n)^2}\right)^2}}$$

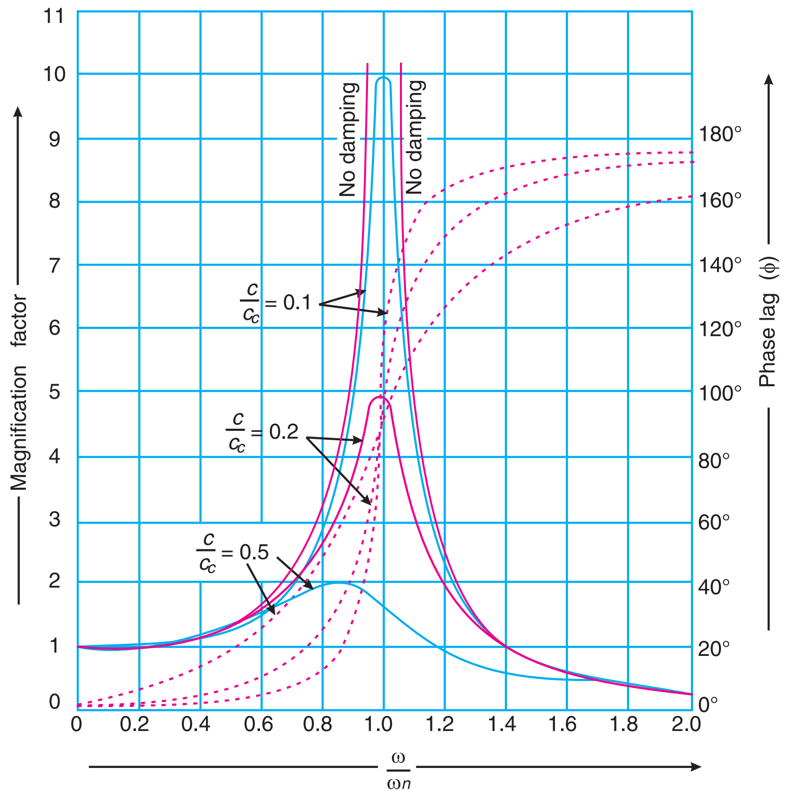


Fig. 23.21. Relationship between magnification factor and phase angle for different values of ω / ω_n.

$\therefore$ Magnification factor or dynamic magnifier,

$$D = \frac{x_{max}}{x_o} = \frac{1}{\sqrt{\frac{c^2.\omega^2}{s^2} + \left(1 - \frac{\omega^2}{(\omega_n)^2}\right)^2}} \qquad \text{... } (i)$$

$$= \frac{1}{\sqrt{\left(\frac{2c.\omega}{c_c.\omega_n}\right)^2 + \left(1 - \frac{\omega^2}{(\omega_n)^2}\right)^2}}$$

Example 5. *A mass of 10 kg is suspended from one end of a helical spring, the other end being fixed. The stiffness of the spring is 10 N/mm. The viscous damping causes the amplitude to decrease to one-tenth of the initial value in four complete oscillations. If a periodic force of 150 cos 50 t N is applied at the mass in the vertical direction, find the amplitude of the forced vibrations. What is its value of resonance ?*

Solution. Given : $m = 10$ kg ; $s = 10$ N/mm $= 10 \times 10^3$ N/m ; $x_5 = \frac{x_1}{10}$

Since the periodic force, $F_x = F \cos \omega t = 150 \cos 50 t$, therefore

Static force, $F = 150$ N

and angular velocity of the periodic disturbing force,

$$\omega = 50 \text{ rad/s}$$

We know that angular speed or natural circular frequency of free vibrations,

$$\omega_n = \sqrt{\frac{s}{m}} = \sqrt{\frac{10 \times 10^3}{10}} = 31.6 \text{ rad/s}$$

Amplitude of the forced vibrations

Since the amplitude decreases to 1/10th of the initial value in four complete oscillations, therefore, the ratio of initial amplitude (x_1) to the final amplitude after four complete oscillations (x_5) is given by

$$\frac{x_1}{x_5} = \frac{x_1}{x_2} \times \frac{x_2}{x_3} \times \frac{x_3}{x_4} \times \frac{x_4}{x_5} = \left(\frac{x_1}{x_2}\right)^4 \qquad \ldots \left(\because \frac{x_1}{x_2} = \frac{x_2}{x_3} = \frac{x_3}{x_4} = \frac{x_4}{x_5}\right)$$

$$\therefore \quad \frac{x_1}{x_2} = \left(\frac{x_1}{x_5}\right)^{1/4} = \left(\frac{x_1}{x_1/10}\right)^{1/4} = (10)^{1/4} = 1.78 \qquad \ldots \left(x_5 = \frac{x_1}{10}\right)$$

We know that

$$\log_e\left(\frac{x_1}{x_2}\right) = a \times \frac{2\pi}{\sqrt{(\omega_n)^2 - a^2}}$$

$$\log_e 1.78 = a \times \frac{2\pi}{\sqrt{(31.6)^2 - a^2}} \quad \text{or} \quad 0.576 = \frac{a \times 2\pi}{\sqrt{1000 - a^2}}$$

Squaring both sides and rearranging,

$$39.832\, a^2 = 332 \quad \text{or} \quad a^2 = 8.335 \quad \text{or} \quad a = 2.887$$

We know that $a = c/2m$ or $c = a \times 2m = 2.887 \times 2 \times 10 = 57.74$ N/m/s

and deflection of the system produced by the static force F,

$$x_o = F/s = 150/10 \times 10^3 = 0.015 \text{ m}$$

We know that amplitude of the forced vibrations,

$$x_{max} = \frac{x_o}{\sqrt{\frac{c^2.\omega^2}{s^2} + \left[1 - \frac{\omega^2}{(\omega_n)^2}\right]^2}}$$

$$= \frac{0.015}{\sqrt{\frac{(57.74)^2 (50)^2}{(10 \times 10^3)^2} + \left[1 - \left(\frac{50}{31.6}\right)^2\right]^2}} = \frac{0.015}{\sqrt{0.083 + 2.25}}$$

$$= \frac{0.015}{1.53} = 9.8 \times 10^{-3} \text{ m} = 9.8 \text{ mm} \quad \textbf{Ans.}$$

Amplitude of forced vibrations at resonance

We know that amplitude of forced vibrations at resonance,

$$x_{max} = x_0 \times \frac{s}{c.\omega_n} = 0.015 \times \frac{10 \times 10^3}{57.54 \times 31.6} = 0.0822 \text{ m} = 82.2 \text{ mm} \quad \textbf{Ans.}$$

Example 6. *A body of mass 20 kg is suspended from a spring which deflects 15 mm under this load. Calculate the frequency of free vibrations and verify that a viscous damping force amounting to approximately 1000 N at a speed of 1 m/s is just-sufficient to make the motion aperiodic. If when damped to this extent, the body is subjected to a disturbing force with a maximum value of 125 N making 8 cycles/s, find the amplitude of the ultimate motion.*

Solution. Given : $m = 20$ kg ; $\delta = 15$ mm $= 0.015$ m ; $c = 1000$ N/m/s ; $F = 125$ N ; $f = 8$ cycles/s

Frequency of free vibrations

We know that frequency of free vibrations,

$$f_n = \frac{1}{2\pi}\sqrt{\frac{g}{\delta}} = \frac{1}{2\pi}\sqrt{\frac{9.81}{0.015}} = 4.07 \text{ Hz} \quad \textbf{Ans.}$$

The critical damping to make the motion aperiodic is such that damped frequency is zero, *i.e.*

$$\left(\frac{c}{2m}\right)^2 = \frac{s}{m}$$

$$\therefore \qquad c = \sqrt{\frac{s}{m} \times 4m^2} = \sqrt{4\,s.m} = \sqrt{4 \times \frac{m.g}{\delta} \times m} \qquad \ldots \left(\because\ s = \frac{m.g}{\delta}\right)$$

$$= \sqrt{4 \times \frac{20 \times 9.81}{0.015} \times 20} = 1023 \text{ N/m/s}$$

This means that the viscous damping force is 1023 N at a speed of 1 m/s. Therefore a viscous damping force amounting to approximately 1000 N at a speed of 1 m/s is just sufficient to make the motion aperiodic. **Ans.**

Amplitude of ultimate motion

We know that angular speed of forced vibration,

$$\omega = 2\pi \times f = 2\pi \times 8 = 50.3 \text{ rad/s}$$

and stiffness of the spring, $s = m.g/\delta = 20 \times 9.81 / 0.015 = 13.1 \times 10^3$ N/m

$\therefore$ Amplitude of ultimate motion *i.e.* maximum amplitude of forced vibration,

$$x_{max} = \frac{F}{\sqrt{c^2.\omega^2 + (s - m.\omega^2)^2}}$$

$$= \frac{125}{\sqrt{(1023)^2 (50.3)^2 + [13.1 \times 10^3 - 20(50.3)^2]^2}}$$

$$= \frac{125}{\sqrt{2600 \times 10^6 + 1406 \times 10^6}} = \frac{125}{63.7 \times 10^3} = 1.96 \times 10^{-3} \text{ m}$$

$= 1.96$ mm **Ans.**

Example 7. *A machine part of mass 2 kg vibrates in a viscous medium. Determine the damping coefficient when a harmonic exciting force of 25 N results in a resonant amplitude of 12.5 mm with a period of 0.2 second. If the system is excited by a harmonic force of frequency 4 Hz what will be the percentage increase in the amplitude of vibration when damper is removed as compared with that with damping.*

Solution. Given : $m = 2$ kg ; $F = 25$ N ; Resonant $x_{max} = 12.5$ mm $= 0.0125$ m ; $t_p = 0.2$ s ; $f = 4$ Hz

Damping coefficient

Let c = Damping coefficient in N/m/s.

We know that natural circular frequency of the exicting force,

$$\omega_n = 2\pi / t_p = 2\pi / 0.2 = 31.42 \text{ rad/s}$$

We also know that the maximum amplitude of vibration at resonance (x_{max}),

$$0.0125 = \frac{F}{c.\omega_n} = \frac{25}{c \times 31.42} = \frac{0.796}{c} \quad \text{or } c = 63.7 \text{ N/m/s} \quad \textbf{Ans.}$$

Percentage increase in amplitude

Since the system is excited by a harmonic force of frequency (f) = 4 Hz, therefore corresponding circular frequency

$$\omega = 2\pi \times f = 2\pi \times 4 = 25.14 \text{ rad/s}$$

We know that maximum amplitude of vibration with damping,

$$x_{max} = \frac{F}{\sqrt{c^2.\omega^2 + (s - m.\omega^2)^2}} = \frac{25}{\sqrt{(63.7)^2 (25.14)^2 + [2(31.42)^2 - 2(25.14)^2]^2}}$$

$$\ldots \left[\because \ (\omega_n)^2 = s/m \quad \text{or} \quad s = m(\omega_n)^2\right]$$

$$= \frac{25}{\sqrt{2.56 \times 10^6 + 0.5 \times 10^6}} = \frac{25}{1749} = 0.0143 \text{ m} = 14.3 \text{ mm}$$

and the maximum amplitude of vibration when damper is removed,

$$x_{max} = \frac{F}{m\left[(\omega_n)^2 - \omega^2\right]} = \frac{25}{2[(31.42)^2 - (25.14)^2]} = \frac{25}{710} = 0.0352 \text{ m}$$

$$= 35.2 \text{ mm}$$

$\therefore$ Percentage increase in amplitude

$$= \frac{35.2 - 14.3}{14.3} = 1.46 \quad \text{or} \quad 146\% \quad \textbf{Ans.}$$

23.18. Vibration Isolation and Transmissibility

A little consideration will show that when an unbalanced machine is installed on the foundation, it produces vibration in the foundation. In order to prevent these vibrations or to minimise the transmission of forces to the foundation, the machines are mounted on springs and dampers or on some vibration isolating material, as shown in Fig. 23.22. The arrangement is assumed to have one degree of freedom, *i.e.* it can move up and down only.

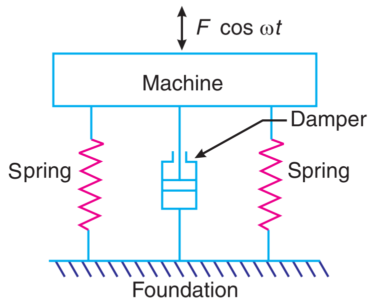


Fig. 23.22. Vibration isolation.

It may be noted that when a periodic (*i.e.* simple harmonic) disturbing force $F \cos \omega t$ is applied to a machine of mass m supported by a spring of stiffness s, then the force is transmitted by means of the spring and the damper or dashpot to the fixed support or foundation.

The ratio of the force transmitted (F_T) to the force applied (F) is known as the ***isolation factor*** or ***transmissibility ratio*** of the spring support.

We have discussed above that the force transmitted to the foundation consists of the following two forces :

1. Spring force or elastic force which is equal to $s.\,x_{max}$, and

2. Damping force which is equal to $c.\,\omega.x_{max}$.

Since these two forces are perpendicular to one another, as shown in Fig.23.23, therefore the force transmitted,

$$F_T = \sqrt{(s.x_{max})^2 + (c.\omega.x_{max})^2}$$

$$= x_{max}\sqrt{s^2 + c^2.\omega^2}$$

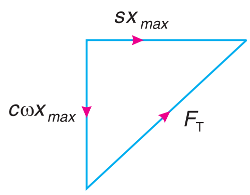


Fig. 23.23

$\therefore$ Transmissibility ratio,

$$\varepsilon = \frac{F_T}{F} = \frac{x_{max}\sqrt{s^2 + c^2.\omega^2}}{F}$$

$$\varepsilon = \frac{\sqrt{1 + \left(\frac{2c.\omega}{c_c.\omega_n}\right)^2}}{\sqrt{\left(\frac{2c.\omega}{c_c.\omega_n}\right)^2 + \left(1 - \frac{\omega^2}{(\omega_n)^2}\right)^2}} \qquad \ldots (i)$$

When the damper is not provided, then $c = 0$, and

$$\varepsilon = \frac{1}{1 - (\omega/\omega_n)^2} \qquad \ldots (ii)$$

Example 8. *The mass of an electric motor is 120 kg and it runs at 1500 r.p.m. The armature mass is 35 kg and its C.G. lies 0.5 mm from the axis of rotation. The motor is mounted on five springs of negligible damping so that the force transmitted is one-eleventh of the impressed force. Assume that the mass of the motor is equally distributed among the five springs.*

*Determine : **1.** stiffness of each spring; **2.** dynamic force transmitted to the base at the operating speed; and **3.** natural frequency of the system.*

Solution. Given m_1 = 120 kg ; m_2 = 35 kg; r = 0.5 mm = 5×10^{-4} m; ε = 1 / 11; N = 1500 r.p.m. or ω = 2π × 1500 / 60 = 157.1 rad/s ;

1. *Stiffness of each spring*

Let s = Combined stiffness of the spring in N-m, and

ω_n = Natural circular frequency of vibration of the machine in rad/s.

We know that transmissibility ratio (ε),

$$\frac{1}{11} = \frac{1}{\left(\dfrac{\omega}{\omega_n}\right)^2 - 1} = \frac{(\omega_n)^2}{\omega^2 - (\omega_n)^2} = \frac{(\omega_n)^2}{(157.1)^2 - (\omega_n)^2}$$

or $$(157.1)^2 - (\omega_n)^2 = 11(\omega_n)^2 \quad \text{or} \quad (\omega_n)^2 = 2057 \quad \text{or} \quad \omega_n = 45.35 \text{ rad/s}$$

We know that $$\omega_n = \sqrt{s / m_1}$$

$$s = m_1 (\omega_n)^2 = 120 \times 2057 = 246\ 840 \text{ N/m}$$

Since these are five springs, therefore stiffness of each spring

= 246 840 / 5 = 49 368 N/m **Ans.**

2. *Dynamic force transmitted to the base at the operating speed (i.e. 1500 r.p.m. or 157.1 rad/s)*

We know that maximum unbalanced force on the motor due to armature mass,

$$F = m_2\, \omega^2 \cdot r = 35 (157.1)^2\, 5 \times 10^{-4} = 432 \text{ N}$$

∴ Dynamic force transmitted to the base,

$$F_T = \varepsilon . F = \frac{1}{11} \times 432 = 39.27 \text{ N} \quad \textbf{Ans.}$$

3. *Natural frequency of the system*

We have calculated above that the natural frequency of the system,

$$\omega_n = 45.35 \text{ rad/s} \quad \textbf{Ans.}$$

UNIT-VI

TRANSVERSE AND TORSIONAL VIBRATIONS

23.8. Natural Frequency of Free Transverse Vibrations Due to a Point Load Acting Over a Simply Supported Shaft

If δ is the static deflection due to load W, then the natural frequency of the free transverse vibration is

$$f_n = \frac{1}{2\pi}\sqrt{\frac{g}{\delta}} = \frac{0.4985}{\sqrt{\delta}} \text{ Hz}$$

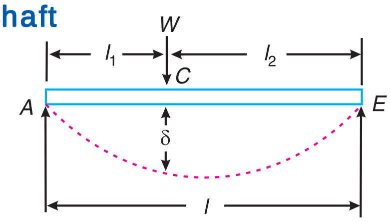
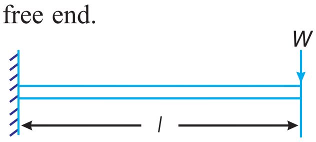
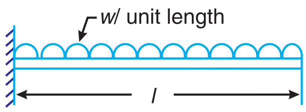
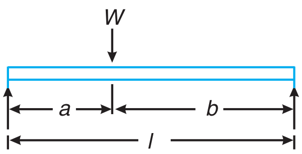
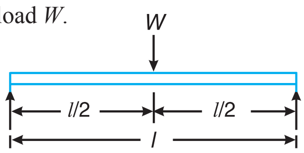
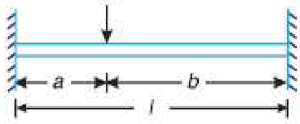
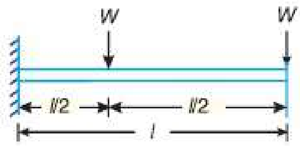


Fig. 23.6. Simply supported beam with a point load.

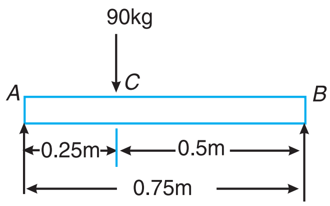
Table 23.1. Values of static deflection (δ) for the various types of beams and under various load conditions.

S.No.	*Type of beam*	*Deflection* (δ)
1.	Cantilever beam with a point load W at the free end. 	$\delta = \frac{Wl^3}{3EI}$ (at the free end)
2.	Cantilever beam with a uniformly distributed load of w per unit length. 	$\delta = \frac{wl^4}{8EI}$ (at the free end)
3.	Simply supported beam with an eccentric point load W. 	$\delta = \frac{Wa^2b^2}{3EIl}$ (at the point load)
4.	Simply supported beam with a central point load W. 	$\delta = \frac{Wl^3}{48EI}$ (at the centre)
5.	Simply supported beam with a uniformly distributed load of w per unit length. 	$\delta = \frac{5}{384}\times\frac{wl^4}{EI}$ (at the centre)

S.No.	Type of beam	Deflection (δ)
6.	Fixed beam with an eccentric point load W. 	$\delta = \dfrac{Wa^3b^3}{3EIl}$ (at the point load)
7.	Fixed beam with a central point load W. 	$\delta = \dfrac{Wl^3}{192EI}$ (at the centre)
8.	Fixed beam with a uniformly distributed load of w per unit length. 	$\delta = \dfrac{wl^4}{384EI}$ (at the centre)

Example 1. *A shaft of length 0.75 m, supported freely at the ends, is carrying a body of mass 90 kg at 0.25 m from one end. Find the natural frequency of transverse vibration. Assume E = 200 GN/m² and shaft diameter = 50 mm.*

Solution. Given : $l = 0.75$ m ; $m = 90$ kg ; $a = AC = 0.25$ m ; $E = 200$ GN/m^2 = 200×10^9 N/m^2; $d = 50$ mm = 0.05 m



The shaft is shown in Fig. 23.7.

We know that moment of inertia of the shaft,

$$I = \frac{\pi}{64} \times d^4 = \frac{\pi}{64}(0.05)^4 \text{ m}^4$$

$$= 0.307 \times 10^{-6} \text{ m}^4$$

and static deflection at the load point (*i.e.* at point C),

$$\delta = \frac{Wa^2b^2}{3EIl} = \frac{90 \times 9.81(0.25)^2(0.5)^2}{3 \times 200 \times 10^9 \times 0.307 \times 10^{-6} \times 0.75} = 0.1 \times 10^{-3} \text{ m}$$

. . . ($\because b = BC = 0.5$ m)

We know that natural frequency of transverse vibration,

$$f_n = \frac{0.4985}{\sqrt{\delta}} = \frac{0.4985}{\sqrt{0.1 \times 10^{-3}}} = 49.85 \text{ Hz} \quad \textbf{Ans.}$$

23.9. Natural Frequency of Free Transverse Vibrations Due to Uniformly Distributed Load Acting Over a Simply Supported Shaft

Consider a shaft AB carrying a uniformly distributed load of w per unit length as shown in Fig. 23.9.

Let y_1 = Static deflection at the middle of the shaft,

a_1 = Amplitude of vibration at the middle of the shaft, and

w_1 = Uniformly distributed load per unit static deflection at the middle of the shaft = w/y_1.

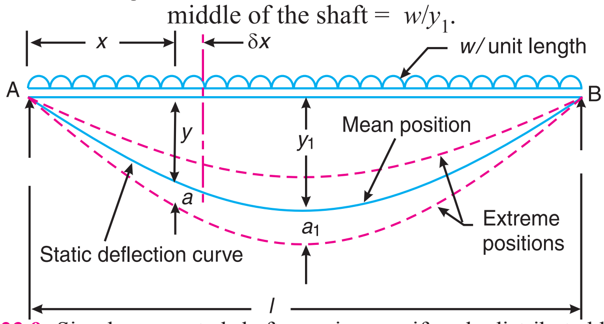


Fig. 23.9. Simply supported shaft carrying a uniformly distributed load.

Now, consider a small section of the shaft at a distance x from A and length δx.

Let y = Static deflection at a distance x from A, and

a = Amplitude of its vibration.

$\therefore$ Work done on this small section

$$= \frac{1}{2} \times w_1 . a_1 . \delta x \times a = \frac{1}{2} \times \frac{w}{y_1} \times a_1 . \delta x \times a = \frac{1}{2} \times w \times \frac{a_1}{y_1} \times a \times \delta x$$

Since the maximum potential energy at the extreme position is equal to the amount of work done to move the beam from the mean position to one of its extreme positions, therefore

Maximum potential energy at the extreme position

$$= \int_0^l \frac{1}{2} \times w \times \frac{a_1}{y_1} \times a . dx \qquad \ldots (i)$$

Assuming that the shape of the curve of a vibrating shaft is similar to the static deflection curve of a beam, therefore

$$\frac{a_1}{y_1} = \frac{a}{y} = \text{Constant, } C \quad \text{or} \quad \frac{a_1}{y_1} = C \ \text{ and } a = y.C$$

Substituting these values in equation ***(i)***, we have maximum potential energy at the extreme position

$$= \int_0^l \frac{1}{2} \times w \times C \times y.C.dx = \frac{1}{2} \times w.C^2 \int_0^l y.dx \qquad \ldots (ii)$$

Since the maximum velocity at the mean position is $\omega . a_1$, where ω is the circular frequency of vibration, therefore

Maximum kinetic energy at the mean position

$$= \int_0^l \frac{1}{2} \times \frac{w.dx}{g} (\omega . a)^2 = \frac{w}{2g} \times \omega^2 \times C^2 \int_0^l y^2 .dx \qquad \ldots (iii)$$

. . .(Substituting $a = y.C$)

We know that the maximum potential energy at the extreme position is equal to the maximum kinetic energy at the mean position, therefore equating equations ***(ii)*** and ***(iii)***,

$$\frac{1}{2} \times w \times C^2 \int_0^l y.dx = \frac{w}{2g} \times \omega^2 \times C^2 \int_0^l y^2 .dx$$

$$\therefore \quad \omega^2 = \frac{g \int_0^l y.dx}{\int_0^l y^2 .dx} \quad \text{or} \quad \omega = \sqrt{\frac{g \int_0^l y.dx}{\int_0^l y^2 .dx}} \qquad \ldots (iv)$$

When the shaft is a simply supported, then the static deflection at a distance x from A is

$$y = \frac{w}{24\, EI} (x^4 - 2l\, x^3 + l^3 x) \qquad \ldots (v)$$

where w = Uniformly distributed load unit length,

E = Young's modulus for the material of the shaft, and

I = Moment of inertia of the shaft.

Now integrating the above equation *(v)* within the limits from 0 to l,

$$\int_0^l y\, dx = \frac{w}{24\, EI} \int_0^l (x^4 - 2\, lx^3 + l^3 x)\, dx = \frac{w.l^5}{120\, E.I} \qquad \ldots (vi)$$

$$\int_0^l y^2 dx = \int_0^l \left[\frac{w}{24\,EI}(x^4 - 2l\,x^3 + l^3 x)\right]^2 dx$$

$$= \frac{w^2}{576\,E^2\,I^2} \times \frac{31\,l^9}{630} \qquad \ldots (vii)$$

Substituting the value in equation **(*iv*)** from equations **(*vi*)** and **(*vii*),** we get circular frequency due to uniformly distributed load,

$$\omega = \sqrt{g\left(\frac{wl^5}{120\,EI} \times \frac{576\,E^2 I^2 \times 630}{w^2 \times 31\,l^9}\right)}$$

$$= \sqrt{\frac{24\,EI}{wl^4} \times \frac{630}{155}\,g} = \pi^2 \sqrt{\frac{EI\,g}{wl^4}} \qquad \ldots (viii)$$

$\therefore$ Natural frequency due to uniformly distributed load,

$$f_n = \frac{\omega}{2\pi} = \frac{\pi^2}{2\pi}\sqrt{\frac{EI\,g}{wl^4}} = \frac{\pi}{2}\sqrt{\frac{EIg}{wl^4}} \qquad \ldots (ix)$$

We know that the static deflection of a simply supported shaft due to uniformly distributed load of w per unit length, is

$$\delta_S = \frac{5\,wl^4}{384\,EI} \quad \text{or} \quad \frac{EI}{wl^4} = \frac{5}{384\,\delta_S}$$

Equation **(*ix*)** may be written as

$$f_n = \frac{\pi}{2}\sqrt{\frac{5\,g}{384\,\delta_S}} = \frac{0.5615}{\sqrt{\delta_S}} \text{ Hz} \qquad \ldots (\text{Substituting, } g = 9.81 \text{ m/s}^2)$$

23.10. Natural Frequency of Free Transverse Vibrations of a Shaft Fixed at Both Ends Carrying a Uniformly Distributed Load

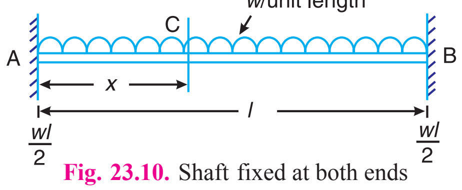


Fig. 23.10. Shaft fixed at both ends carrying a uniformly distributed load.

Consider a shaft AB fixed at both ends and carrying a uniformly distributed load of w per unit length as shown in Fig. 23.10.

We know that the static deflection at a distance x from A is given by

$$* \quad y = \frac{w}{24\,EI}(x^4 + l^2 x^2 - 2lx^3) \qquad \ldots (i)$$

Integrating the above equation within limits from 0 to l,

$$* \quad \int_0^l y\,dx = \frac{w}{24\,EI}\int_0^l (x^4 + l^2 x^2 - 2l\,x^3)\,dx = \frac{w}{24\,EI} \times \frac{l^5}{30} = \frac{wl^5}{720\,EI}$$

Now integrating y^2 within the limits from 0 to l,

$$\int_0^l y^2\,dx = \left(\frac{w}{24\,EI}\right)^2 \int_0^l (x^4 + l^2 x^2 - 2l\,x^3)^2\,dx = \left(\frac{w}{24\,EI}\right)^2 \frac{l^9}{630}$$

We know that $\qquad \omega^2 = \dfrac{g\displaystyle\int_0^l y\,dx}{\displaystyle\int_0^l y^2\,dx} = \dfrac{504\,EIg}{wl^4}$

Since the static deflection of a shaft fixed at both ends and carrying a uniformly distributed load is

$$\delta_S = \frac{wl^4}{384\,EI}$$

and natural frequency, $\qquad f_n = \dfrac{\omega}{2\pi} = \dfrac{1}{2\pi}\sqrt{\dfrac{504\,EIg}{wl^4}} = 3.573\sqrt{\dfrac{E\,I\,g}{w\,l^4}} = \dfrac{0.571}{\sqrt{\delta_S}}$ Hz

23.11. Natural Frequency of Free Transverse Vibrations For a Shaft Subjected to a Number of Point Loads

Consider a shaft AB of negligible mass loaded with point loads W_1 , W_2, W_3 and W_4 etc. in newtons, as shown in Fig. 23.11. Let m_1, m_2, m_3 and m_4 etc. be the corresponding masses in kg. The natural frequency of such a shaft may be found out by the following two methods :

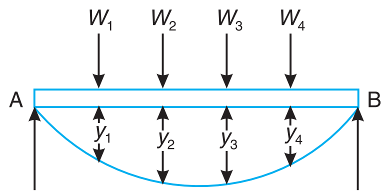


Fig. 23.11. Shaft carrying a number of point loads.

1. Energy (or Rayleigh's) method

Let y_1, y_2, y_3, y_4 etc. be total deflection under loads W_1, W_2, W_3 and W_4 etc. as shown in Fig. 23.11.

We know that maximum potential energy

$$= \frac{1}{2} \times m_1.g.y_1 + \frac{1}{2} \times m_2.g.y_2 + \frac{1}{2} m_3.g.y_3 + \frac{1}{2} \times m_4.g.y_4 + ... = \frac{1}{2} \Sigma m.g.y$$

and maximum kinetic energy

$$= \frac{1}{2} \times m_1 (\omega.y_1)^2 + \frac{1}{2} \times m_2 (\omega.y_2)^2 + \frac{1}{2} \times m_3 (\omega.y_3)^2 + \frac{1}{2} \times m_4 (\omega.y_4)^2 + ...$$

$$= \frac{1}{2} \times \omega^2 \left[m_1 (y_1)^2 + m_2 (y_2)^2 + m_3 (y_3)^2 + m_4 (y_4)^2 + ... \right]$$

$$= \frac{1}{2} \times \omega^2 \Sigma m.y^2 \qquad \text{. . . (where } \omega = \text{Circular frequency of vibration)}$$

Equating the maximum kinetic energy to the maximum potential energy, we have

$$\frac{1}{2} \times \omega^2 \Sigma m.y^2 = \frac{1}{2} \Sigma m.g.y$$

$$\therefore \qquad \omega^2 = \frac{\Sigma m.g.y}{\Sigma m.y^2} = \frac{g \Sigma m.y}{\Sigma m.y^2} \qquad \text{or} \qquad \omega = \sqrt{\frac{g \Sigma m.y}{\Sigma m.y^2}}$$

$\therefore$ Natural frequency of transverse vibration,

$$f_n = \frac{\omega}{2\pi} = \frac{1}{2\pi} \sqrt{\frac{g \Sigma m.y}{\Sigma m.y^2}}$$

2. Dunkerley's method

The natural frequency of transverse vibration for a shaft carrying a number of point loads and uniformly distributed load is obtained from Dunkerley' s empirical formula. According to this

$$\frac{1}{(f_n)^2} = \frac{1}{(f_{n1})^2} + \frac{1}{(f_{n2})^2} + \frac{1}{(f_{n3})^2} + ... + \frac{1}{(f_{ns})^2}$$

where f_n = Natural frequency of transverse vibration of the shaft carrying point loads and uniformly distributed load.

f_{n1}, f_{n2}, f_{n3}, etc. = Natural frequency of transverse vibration of each point load.

f_{ns} = Natural frequency of transverse vibration of the uniformly distributed load (or due to the mass of the shaft).

Now, consider a shaft AB loaded as shown in Fig. 23.12.

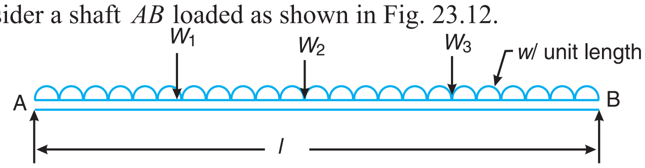


Fig. 23.12. Shaft carrying a number of point loads and a uniformly distributed load.

Let $\delta_1, \delta_2, \delta_3,$ etc. = Static deflection due to the load $W_1,\ W_2,\ W_3$ etc. when considered separately.

δ_S = Static deflection due to the uniformly distributed load or due to the mass of the shaft.

We know that natural frequency of transverse vibration due to load W_1,

$$f_{n_1} = \frac{0.4985}{\sqrt{\delta_1}} \text{ Hz}$$

Similarly, natural frequency of transverse vibration due to load W_2,

$$f_{n_2} = \frac{0.4985}{\sqrt{\delta_2}} \text{ Hz}$$

and, natural frequency of transverse vibration due to load W_3,

$$f_{n_3} = \frac{0.4985}{\sqrt{\delta_3}} \text{ Hz}$$

Also natural frequency of transverse vibration due to uniformly distributed load or weight of the shaft,

$$f_{ns} = \frac{0.5615}{\sqrt{\delta_S}} \text{ Hz}$$

Therefore, according to Dunkerley's empirical formula, the natural frequency of the whole system,

$$\frac{1}{(f_n)^2} = \frac{1}{(f_{n1})^2} + \frac{1}{(f_{n2})^2} + \frac{1}{(f_{n3})^3} + \ldots + \frac{1}{(f_{ns})^2}$$

$$= \frac{\delta_1}{(0.4985)^2} + \frac{\delta_2}{(0.4985)^2} + \frac{\delta_3}{(0.4985)^2} + \ldots + \frac{\delta_S}{(0.5615)^2}$$

$$= \frac{1}{(0.4985)^2}\left[\delta_1 + \delta_2 + \delta_3 + \ldots + \frac{\delta_S}{1.27}\right]$$

or

$$f_n = \frac{0.4985}{\sqrt{\delta_1 + \delta_2 + \delta_3 + \ldots + \frac{\delta_S}{1.27}}} \text{ Hz}$$

Example 1. *A shaft 50 mm diameter and 3 metres long is simply supported at the ends and carries three loads of 1000 N, 1500 N and 750 N at 1 m, 2 m and 2.5 m from the left support. The Young's modulus for shaft material is 200 GN/m². Find the frequency of transverse vibration.*

Solution. Given : d = 50 mm = 0.05 m ; l = 3 m, W_1 = 1000 N ; W_2 = 1500 N ; W_3 = 750 N; $E = 200 \text{ GN/m}^2 = 200 \times 10^9 \text{ N/m}^2$

The shaft carrying the loads is shown in Fig. 23.13

We know that moment of inertia of the shaft,

$$I = \frac{\pi}{64} \times d^4 = \frac{\pi}{64}(0.05)^4 = 0.307 \times 10^{-6} \text{ m}^4$$

and the static deflection due to a point load W,

$$\delta = \frac{Wa^2b^2}{3EIl}$$

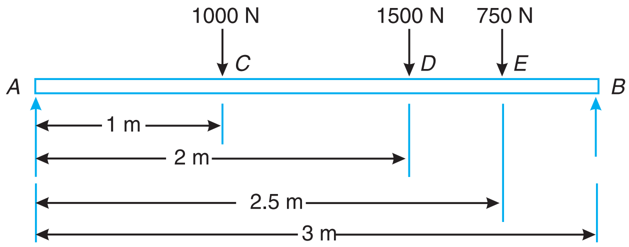


Fig. 23.13

$\therefore$ Static deflection due to a load of 1000 N,

$$\delta_1 = \frac{1000 \times 1^2 \times 2^2}{3 \times 200 \times 10^9 \times 0.307 \times 10^{-6} \times 3} = 7.24 \times 10^{-3} \text{ m}$$

. . . (Here $a = 1$ m, and $b = 2$ m)

Similarly, static deflection due to a load of 1500 N,

$$\delta_2 = \frac{1500 \times 2^2 \times 1^2}{3 \times 200 \times 10^9 \times 0.307 \times 10^{-6} \times 3} = 10.86 \times 10^{-3} \text{ m}$$

. . . (Here $a = 2$ m, and $b = 1$ m)

and static deflection due to a load of 750 N,

$$\delta_3 = \frac{750\,(2.5)^2\,(0.5)^2}{3 \times 200 \times 10^9 \times 0.307 \times 10^{-6} \times 3} = 2.12 \times 10^{-3} \text{ m}$$

. . . (Here $a = 2.5$ m, and $b = 0.5$ m)

We know that frequency of transverse vibration,

$$f_n = \frac{0.4985}{\sqrt{\delta_1 + \delta_2 + \delta_3}} = \frac{0.4985}{\sqrt{7.24 \times 10^{-3} + 10.86 \times 10^{-3} + 2.12 \times 10^{-3}}}$$

$$= \frac{0.4985}{0.1422} = 3.5 \text{ Hz } \textbf{Ans.}$$

23.12. Critical or Whirling Speed of a Shaft

The speed at which the shaft runs so that the additional deflection of the shaft from the axis of rotation becomes infinite, is known as *critical* or *whirling speed*.

$\therefore$ Critical or whirling speed,

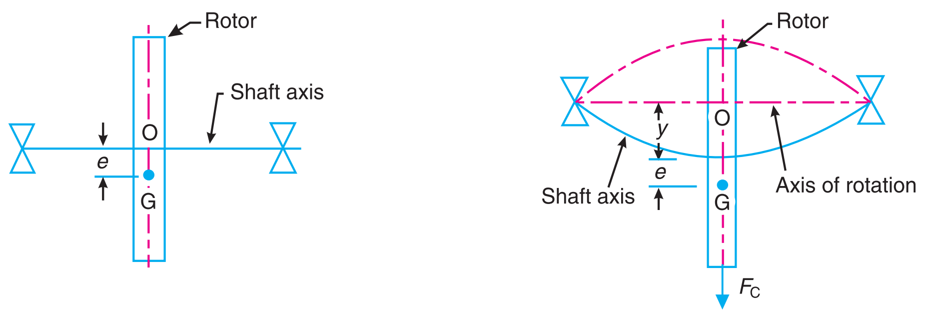
$$\omega_c = \omega_n = \sqrt{\frac{s}{m}} = \sqrt{\frac{g}{\delta}} \text{ Hz} \qquad \ldots \left(\because\ \delta = \frac{m.g}{s}\right)$$

If N_c is the critical or whirling speed in r.p.s., then

$$2\pi N_c = \sqrt{\frac{g}{\delta}} \quad \text{or} \quad N_c = \frac{1}{2\pi}\sqrt{\frac{g}{\delta}} = \frac{0.4985}{\sqrt{\delta}} \text{ r.p.s.}$$

where δ = Static deflection of the shaft in metres.

Hence the **critical or whirling speed is the same as the natural frequency of transverse vibration but its unit will be revolutions per second.**



(*a*) When shaft is stationary. (*b*) When shaft is rotating.

Fig. 2

Example 2. *A shaft 1.5 m long, supported in flexible bearings at the ends carries two wheels each of 50 kg mass. One wheel is situated at the centre of the shaft and the other at a distance of 375 mm from the centre towards left. The shaft is hollow of external diameter 75 mm and internal diameter 40 mm. The density of the shaft material is 7700 kg/m^3 and its modulus of elasticity is 200 GN/m^2. Find the lowest whirling speed of the shaft, taking into account the mass of the shaft.*

Solution. l = 1.5 m ; $m_1 = m_2$ = 50 kg ; d_1 = 75 mm = 0.075 m ; d_2 = 40 mm = 0.04 m ; ρ = 7700 kg/m^3 ; E = 200 GN/m^2 = 200 × 10^9 N/m^2

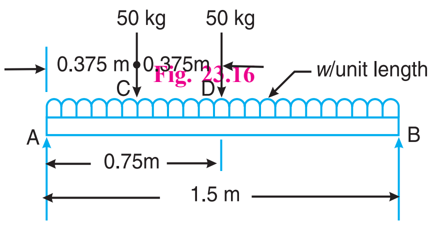


Fig. 23.16

The shaft is shown in Fig. 23.16.

We know that moment of inertia of the shaft,

$$I = \frac{\pi}{64}\left[(d_1)^4 - (d_2)^4\right] = \frac{\pi}{64}\left[(0.075)^4 - (0.04)^4\right] = 1.4 \times 10^{-6}\,\text{m}^4$$

Since the density of shaft material is 7700 kg/m^3, therefore mass of the shaft per metre length,

$$m_S = \text{Area} \times \text{length} \times \text{density}$$

$$= \frac{\pi}{4}\left[(0.075)^2 - (0.04)^2\right]1 \times 7700 = 24.34 \text{ kg/m}$$

We know that the static deflection due to a load W

$$= \frac{Wa^2b^2}{3EIl} = \frac{m.ga^2b^2}{3EIl}$$

∴ Static deflection due to a mass of 50 kg at C,

$$\delta_2 = \frac{m_1 ga^2b^2}{3EIl} = \frac{50 \times 9.81(0.75)^2(0.75)^2}{3 \times 200 \times 10^9 \times 1.4 \times 10^{-6} \times 1.5} = 123 \times 10^{-6} \text{ m}$$

. . . (Here $a = b$ = 0.75 m)

$$\delta_1 = \frac{m_1 ga^2b^2}{3EIl} = \frac{50 \times 9.81(0.375)^2(1.125)^2}{3 \times 200 \times 10^9 \times 1.4 \times 10^{-6} \times 1.5} = 70 \times 10^{-6} \text{ m}$$

. . . (Here a = 0.375 m, and b = 1.125 m)

- Similarly, static deflection due to a mass of 50 kg at D

We know that static deflection due to uniformly distributed load or mass of the shaft,

$$\delta_S = \frac{5}{384} \times \frac{wl^4}{EI} = \frac{5}{384} \times \frac{24.34 \times 9.81(1.5)^4}{200 \times 10^9 \times 1.4 \times 10^{-6}} = 56 \times 10^{-6} \text{ m}$$

. . . (Substituting, $w = m_S \times g$)

We know that frequency of transverse vibration,

$$f_n = \frac{0.4985}{\sqrt{\delta_1 + \delta_2 + \dfrac{\delta_S}{1.27}}} = \frac{0.4985}{\sqrt{70 \times 10^{-6} + 123 \times 10^{-6} + \dfrac{56 \times 10^{-6}}{1.27}}} \text{ Hz}$$

$$= 32.4 \text{ Hz}$$

Since the whirling speed of shaft (N_c) in r.p.s. is equal to the frequency of transverse vibration in Hz, therefore

$$N_c = 32.4 \text{ r.p.s.} = 32.4 \times 60 = 1944 \text{ r.p.m.}$$ **Ans.**

TORSIONAL VIBRATIONS

24.1. Introduction

The particles of a shaft or disc move in a circle about the axis of a shaft, then the vibrations are known as ***torsional vibrations.***

24.2. Natural Frequency of Free Torsional Vibrations

Consider a shaft of negligible mass whose one end is fixed and the other end carrying a disc as shown in Fig. 24.1.

Let θ = Angular displacement of the shaft from mean position after time t in radians,

m = Mass of disc in kg,

I = Mass moment of inertia of disc in kg-m^2 $= m.k^2$,

k = Radius of gyration in metres,

q = Torsional stiffness of the shaft in N-m.

$\therefore$ Restoring force $= q.\theta$ ... ***(i)***

and accelerating force $= I \times \dfrac{d^2\theta}{dt^2}$ ... ***(ii)***

Equating equations ***(i)*** and ***(ii)***, the equation of motion is

$$I \times \frac{d^2\theta}{dt^2} = -q.\theta$$

or

$$\frac{d^2\theta}{dt^2} + \frac{q}{I} \times \theta = 0 \qquad \text{. . . } (iii)$$

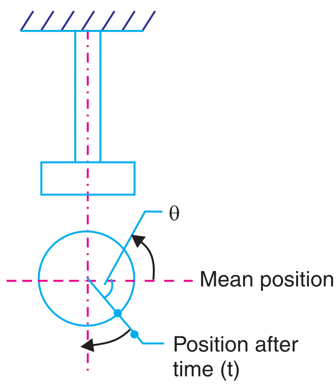


Fig 24.1. Natural frequency of free torsional vibrations.

The fundamental equation of the simple harmonic motion is

$$\frac{d^2\theta}{dt^2} + \omega^2.x = 0 \qquad \text{. . . } (iv)$$

Comparing equations ***(iii)*** and ***(iv)***,

$$\omega = \sqrt{\frac{q}{I}}$$

$\therefore$ Time period, $$t_p = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{I}{q}}$$

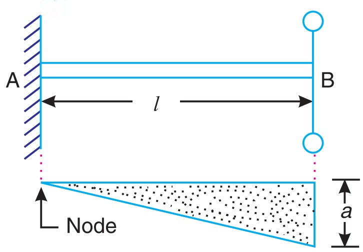
and natural frequency , $$f_n = \frac{1}{t_p} = \frac{1}{2\pi}\sqrt{\frac{q}{I}}$$

24.4. Free Torsional Vibrations of a Single Rotor System

We have already discussed that for a shaft fixed at one end and carrying a rotor at the free end as shown in Fig. 24.4, the natural frequency of torsional vibration,

$$f_n \quad \frac{1}{2}\sqrt{\frac{q}{I}} \quad \frac{1}{2}\sqrt{\frac{C.J}{l.I}}$$

$$\text{. . .} \qquad q \quad \frac{C.J}{l}$$



where C = Modulus of rigidity for shaft material,

J = Polar moment of inertia of shaft

24.5. Free Torsional Vibrations of a Two Rotor System

Consider a two rotor system as shown in Fig. 24.5. It consists of a shaft with two rotors at its ends. In this system, the torsional vibrations occur only when the two rotors *A* and *B* move in opposite directions *i.e.* if *A* moves in anticlockwise direction then *B* moves in clockwise direction at the same instant and *vice versa*. It may be noted that the two rotors must have the same frequency.

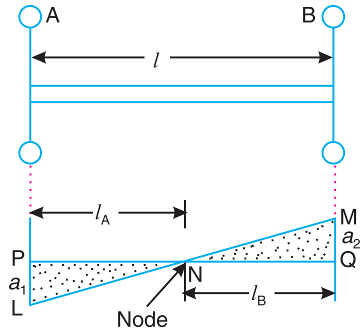


Fig 24.5. Free torsional vibrations of a two rotor system.

We see from Fig. 24.5 that the node lies at point *N*. This point can be safely assumed as a fixed end and the shaft may be considered as two separate shafts *N P* and *N Q* each fixed to one of its ends and carrying rotors at the free ends.

Let l = Length of shaft,

l_A = Length of part *NP* *i.e.* distance of node from rotor *A*,

l_B = Length of part *NQ*, *i.e.* distance of node from rotor *B*,

I_A = Mass moment of inertia of rotor *A*,

I_B = Mass moment of inertia of rotor *B*,

d = Diameter of shaft,

J = Polar moment of inertia of shaft, and

C = Modulus of rigidity for shaft material.

Natural frequency of torsional vibration for rotor *A*,

$$f_{nA} \quad \frac{1}{2}\sqrt{\frac{C.J}{l_A.I_A}} \qquad \ldots (i)$$

and natural frequency of torsional vibration for rotor *B*,

$$f_{nB} \quad \frac{1}{2}\sqrt{\frac{C.J}{l_B.I_B}} \qquad \ldots (ii)$$

Since $f_{nA} = f_{nB}$, therefore

$$\frac{1}{2}\sqrt{\frac{C.J}{l_A.I_A}} \quad \frac{1}{2}\sqrt{\frac{C.J}{l_B.I_B}} \quad \text{or} \quad l_A \cdot I_A = l_B \cdot I_B \qquad \ldots (iii)$$

$$l_A \quad \frac{l_B.I_B}{I_A}$$

We also know that

$$l \quad l_A \quad I_B \qquad \ldots (iv)$$

From equations ***(iii)*** and ***(iv)***, we may find the value of l_A and l_B and hence the position of node. Substituting the values of l_A or l_B in equation ***(i)*** or ***(ii)***, the natural frequency of torsional vibration for a two rotor system may be evaluated.

Note : The line *LNM* in Fig.24.5 is known as **elastic line** for the shaft.

24.6. Free Torsional Vibrations of a Three Rotor System

Consider a three rotor system as shown is Fig. 24.6 (*a*). It consists of a shaft and three rotors *A, B* and *C*. The rotors *A* and *C* are attached to the ends of a shaft, whereas the rotor *B* is attached in between *A* and *C*. The torsional vibrations may occur in two ways, that is with either one node or two nodes. In each case, the two rotors rotate in one direction and the third rotor rotates in opposite direction with the same frequency. Let the rotors *A* and *C* of the system, as shown in Fig. 24.6 (*a*), rotate in the same direction and the rotor *B* in opposite direction. Let the nodal points or nodes of such a system lies at N_1 and N_2 as shown in Fig. 24.6 (*b*). As discussed in Art. 24.5, the shaft may be assumed as a fixed end at the nodes.

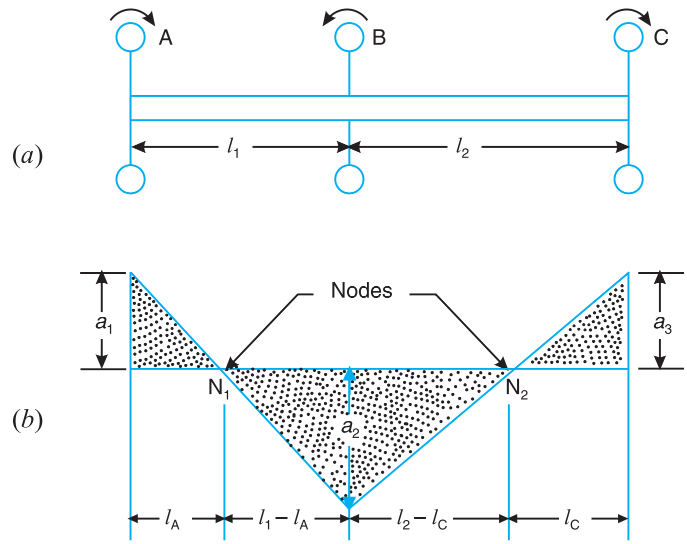


Fig. 24.6. Free torsional vibrations of a three rotor system.

Let l_1 = Distance between rotors *A* and *B*,

l_2 = Distance between rotors *B* and *C*,

l_A = Distance of node N_1 from rotor *A*,

l_C = Distance of node N_2 from rotor *C*,

I_A = Mass moment of inertia of rotor *A*,

I_B = Mass moment of inertia of rotor *B*,

I_C = Mass moment of inertia of rotor *C*,

d = Diameter of shaft,

J = Polar moment of inertia of shaft, and

C = Modulus of rigidity for shaft material.

Natural frequency of torsional vibrations for rotor A,

$$f_{nA} \quad \frac{1}{2}\sqrt{\frac{C.J}{l_A . I_A}} \qquad \ldots (i)$$

Natural frequency of torsional vibrations for rotor B,

$$*f_{nB} \quad \frac{1}{2}\sqrt{\frac{C.J}{I_B}\quad \frac{1}{l_1 \quad l_A} \quad \frac{1}{l_2 \quad l_C}} \qquad \ldots (ii)$$

and natural frequency of torsional vibrations for rotor C,

$$f_{nC} \quad \frac{1}{2}\sqrt{\frac{C.J}{l_C . I_C}} \qquad \ldots (iii)$$

Since $f_{nA} = f_{nB} = f_{nC,}$ therefore equating equations ***(i)*** and ***(iIi)***

$$\frac{1}{2}\sqrt{\frac{C.J}{l_A . I_A}} \quad \frac{1}{2}\sqrt{\frac{C.J}{l_C . I_C}} \qquad \text{or} \qquad l_A . I_A = l_C . I_C$$

$$l_A \quad \frac{l_C . I_C}{I_A} \qquad \ldots (iv)$$

Now equating equations ***(ii)*** and ***(iii)*,**

$$\frac{1}{2}\sqrt{\frac{C.J}{I_B}\quad \frac{1}{l_1 \quad l_A} \quad \frac{1}{l_2 \quad l_C}} \quad \frac{1}{2}\sqrt{\frac{C.J}{l_C . I_C}}$$

or — —— —— ——

24.7. Torsionally Equivalent Shaft

In the previous articles, we have assumed that the shaft is of uniform diameter. But in actual practice, the shaft may have variable diameter for different lengths. Such a shaft may, theoretically, be replaced by an equivalent shaft of uniform diameter.

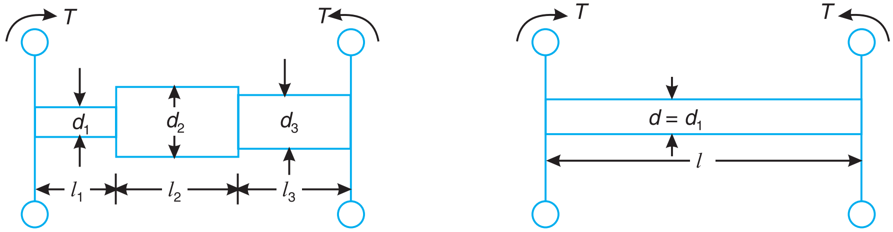
Consider a shaft of varying diameters as shown in Fig. 24.8 (*a*). Let this shaft is replaced by an equivalent shaft of uniform diameter *d* and length *l* as shown in Fig.24.8 (*b*).These two shafts must have the same total angle of twist when equal opposing torques *T* are applied at their opposite ends.

Let d_1, d_2 and d_3 = Diameters for the lengths l_1, l_2 and l_3 respectively,

$_1$, $_2$ and $_3$ = Angle of twist for the lengths l_1, l_2 and l_3 respectively,

= Total angle of twist, and

J_1, J_2 and J_3 = Polar moment of inertia for the shafts of diameters d_1, d_2 and d_3 respectively.



(*a*) Shaft of varying diameters. (*b*) Torsionally equivalent shaft.

Fig 24.8

Since the total angle of twist of the shaft is equal to the sum of the angle of twists of different lengths, therefore

$$_1 \quad _2 \quad _3$$

or

$$\frac{T.l}{C.J} \quad \frac{T.l_1}{C.J_1} \quad \frac{T.l_2}{C.J_2} \quad \frac{T.l_3}{C.J_3}$$

$$\frac{l}{J} \quad \frac{l_1}{J_1} \quad \frac{l_2}{J_2} \quad \frac{l_3}{J_3}$$

$$\frac{l}{\overline{32}\; d^4} \quad \frac{l_1}{\overline{32}(d_1)^4} \quad \frac{l_2}{\overline{32}(d_2)^4} \quad \frac{l_3}{\overline{32}(d_3)^4}$$

$$\frac{l}{d^4} \quad \frac{l_1}{(d_1)^4} \quad \frac{l_2}{(d_2)^4} \quad \frac{l_3}{(d_3)^4}$$

In actual calculations, it is assumed that the diameter d of the equivalent shaft is equal to one of the diameter of the actual shaft. Let us assume that $d = d_1$.

$$\frac{l}{(d_1)^4} \quad \frac{l_1}{(d_1)^4} \quad \frac{l_2}{(d_2)^4} \quad \frac{l_3}{(d_3)^4}$$

or

$$l \quad l_1 \quad l_2 \; \frac{d_1}{d_2}^{\,4} \quad l_3 \; \frac{d_1}{d_3}^{\,4}$$

This expression gives the length l of an equivalent shaft.

Example 24.3. *A steel shaft 1.5 m long is 95 mm in diameter for the first 0.6 m of its length, 60 mm in diameter for the next 0.5 m of the length and 50 mm in diameter for the remaining 0.4 m of its length. The shaft carries two flywheels at two ends, the first having a mass of 900 kg and 0.85 m radius of gyration located at the 95 mm diameter end and the second having a mass of 700 kg and 0.55 m radius of gyration located at the other end. Determine the location of the node and the natural frequency of free torsional vibration of the system. The modulus of rigidity of shaft material may be taken as 80 GN/m*^{2}.

Solution. Given : $L = 1.5$ m ; $d_1 = 95$ mm $= 0.095$ m ; $l_1 = 0.6$ m ; $d_2 = 60$ mm $= 0.06$ m ; $l_2 = 0.5$ m ; $d_3 = 50$ mm $= 0.05$ m ; $l_3 = 0.4$ m ; $m_A = 900$ kg ; $k_A = 0.85$ m ; $m_B = 700$ kg ; $k_B = 0.55$ m ; $C = 80$ GN/m^2 $= 80 \times 10^9$ N/m^2

The actual shaft is shown in Fig. 24.9 (*a*). First of all, let us find the length of the equivalent shaft, assuming its diameter as $d_1 = 95$ mm as shown in Fig 24.9 (*b*).

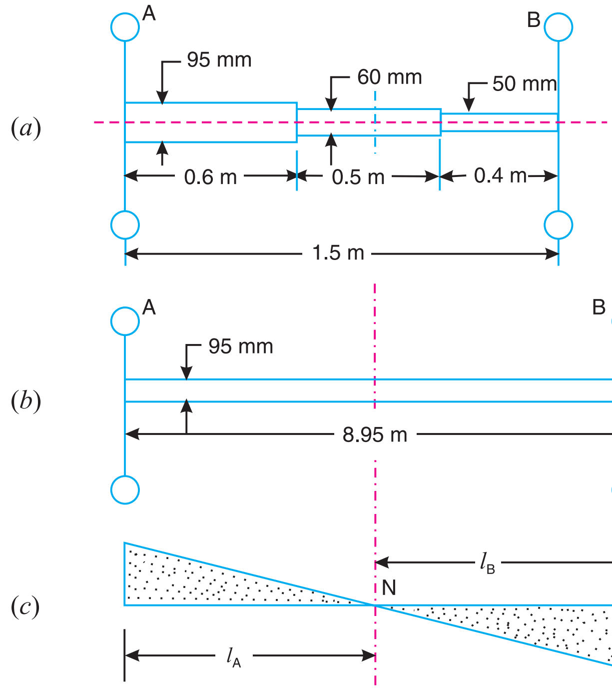


Fig. 24.9

We know that length of the equivalent shaft,

$$l = l_1 + l_2 \left(\frac{d_1}{d_2}\right)^4 + l_3 \left(\frac{d_1}{d_3}\right)^4 = 0.6 + 0.5 \left(\frac{0.095}{0.06}\right)^4 + 0.4 \left(\frac{0.095}{0.05}\right)^4$$

$$= 0.6 + 3.14 + 5.21 = 8.95 \text{ m}$$

Location of the node

Suppose the node of the equivalent shaft lies at N as shown in Fig. 24.9 (*c*).

Let l_A = Distance of the node from flywheel A, and

l_B = Distance of the node from flywheel B.

We know that mass moment of inertia of flywheel A,

$$I_A = m_A (k_A)^2 = 900 (0.85)^2 = 650 \text{ kg-m}^2$$

and mass moment of inertia of flywheel B,

$$I_B = m_B (k_B)^2 = 700 (0.55)^2 = 212 \text{ kg-m}^2$$

We know that $\quad l_A . I_A = l_B . I_B \quad$ or $\quad l_A = \dfrac{l_B . I_B}{I_A} = \dfrac{l_B \times 212}{650} = 0.326 \, l_B$

Also, $\quad l_A + l_B = l = 8.95$ m $\quad$ or $\quad 0.326 \, l_B + l_B = 8.95 \quad$ or $\quad l_B = 6.75$ m

and $\quad l_A = 8.95 - 6.75 = 2.2$ m

Hence the node lies at 2.2 m from flywheel *A* or 6.75 m from flywheel *B* on the equivalent shaft.

∴ Position of node on the original shaft from flywheel *A*

$$l_1 + (l_A - l_1)\left(\frac{d_2}{d_1}\right)^4 = 0.6 + (2.2 - 0.6)\left(\frac{0.06}{0.095}\right)^4 = 0.855 \text{ m} \quad \textbf{Ans.}$$

Natural frequency of free torsional vibrations

We know that polar moment of inertia of the equivalent shaft,

$$J = \frac{\pi}{32}(d_1)^4 = \frac{\pi}{32}(0.095)^4 = 8 \times 10^{-6} \text{ m}^4$$

Natural frequency of free torsional vibrations,

$$f_n = f_{nA} \text{ or } f_{nB}$$

$$= \frac{1}{2\pi}\sqrt{\frac{C.J}{l_A.I_A}} = \frac{1}{2\pi}\sqrt{\frac{80 \times 10^9 \times 8 \times 10^{-6}}{2.2 \times 650}} = 3.37 \text{ Hz} \qquad \textbf{Ans.}$$

Example 24.4. *A steel shaft ABCD 1.5 m long has flywheel at its ends A and D. The mass of the flywheel A is 600 kg and has a radius of gyration of 0.6 m. The mass of the flywheel D is 800 kg and has a radius of gyration of 0.9 m. The connecting shaft has a diameter of 50 mm for the portion AB which is 0.4 m long ; and has a diameter of 60 mm for the portion BC which is 0.5 m long ; and has a diameter of d mm for the portion CD which is 0.6 m long. Determine :*

***1.** the diameter 'd' of the portion CD so that the node of the torsional vibration of the system will be at the centre of the length BC ; and **2.** the natural frequency of the torsional vibrations.*

The modulus of rigidity for the shaft material is 80 GN/m².

Solution. Given : L = 1.5 m ; m_A = 600 kg ; k_A = 0.6 m ; m_D = 800 kg ; k_D = 0.9 m ; d_1 = 50 mm = 0.05 m ; l_1 = 0.4 m ; d_2 = 60 mm = 0.06 m ; l_2 = 0.5 m ; $d_3 = d$; l_3 = 0.6 m ; C = 80 GN/m² = 80×10^9 N/m²

The actual shaft is shown is Fig. 24.10 (a). First of all, let us find the length of the equivalent shaft, assuming its diameter as d_1 = 50 mm, as shown in Fig. 24.10 (b).

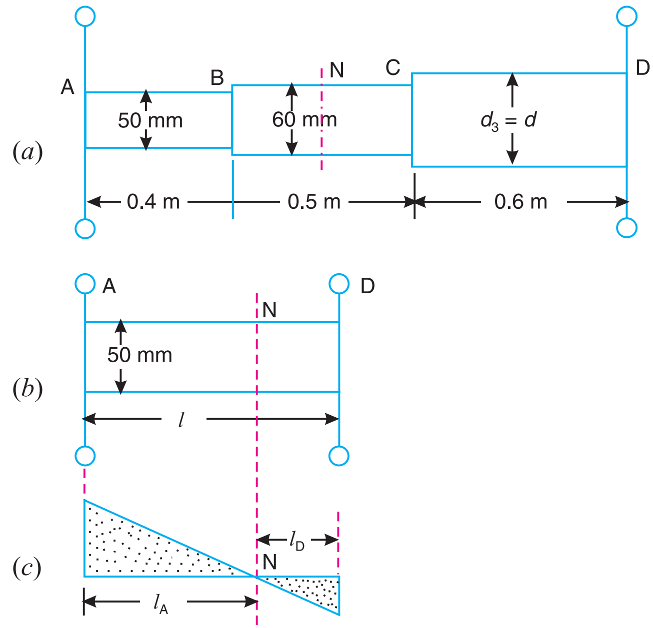


Fig. 24.10

We know that length of the equivalent shaft,

$$l = l_1 + l_2\left(\frac{d_1}{d_2}\right)^4 + l_3\left(\frac{d_1}{d_3}\right)^4 = 0.4 + 0.5\left(\frac{0.05}{0.06}\right)^4 + 0.6\left(\frac{0.05}{d}\right)^4$$

. . . [Substituting $d_3 = d$]

$$= 0.4 + 0.24 + \frac{3.75 \times 10^{-6}}{d^4} = 0.64 + \frac{3.75 \times 10^{-6}}{d^4} \qquad \ldots (i)$$

1. *Diameter 'd' of the shaft CD*

Suppose the node of the equivalent shaft lies at N as shown in Fig. 24.10 (c).

Let l_A = Distance of the node from flywheel A, and

l_D = Distance of the node from flywheel D.

We know that mass moment of inertia of flywheel A,

$$I_A = m_A (k_A)^2 = 600\,(0.6)^2 = 216 \text{ kg-m}^2$$

and mass moment of inertia of flywheel D,

$$I_D = m_D (k_D)^2 = 800\,(0.9)^2 = 648 \text{ kg-m}^2$$

We know that

$$l_A . I_A \quad l_D . I_D$$

or
$$l_D \quad \frac{l_A . I_A}{I_D} \quad \frac{l_A \quad 216}{648} \quad \frac{l_A}{3}$$

Since the node lies in the centre of the length BC in an original system, therefore its equivalent length from rotor A,

$$l_A \quad l_1 \quad \frac{l_2}{2}\frac{d_1}{d_2}^{4} \quad 0.4 \quad \frac{0.5}{2}\frac{0.05}{0.06}^{4} \quad 0.52\,\text{m}$$

$$l_D \quad \frac{l_A}{3} \quad \frac{0.52}{3} = 0.173 \text{ m}$$

We know that $l = l_A + l_D$

or
$$0.64 \quad \frac{3.75 \quad 10^{\quad 6}}{d^4} \quad 0.52 \quad 0.173 \qquad \text{. . . [From equation } (\textit{i})\text{]}$$

$$\frac{3.75 \quad 10^{\quad 6}}{d^4} \quad 0.52 \quad 0.173 \quad 0.64 \quad 0.053$$

$$d^4 \quad \frac{3.75 \quad 10^{\quad 6}}{0.053} \quad 70.75 \quad 10^{\quad 6}$$

or
$$d = 0.0917 \text{ m} = 91.7 \text{ mm} \quad \textbf{Ans.}$$

2. *Natural frequency of torsional vibrations*

We know that polar moment of inertia of the equivalent shaft,

$$J \quad \frac{}{2}(d_1)^4 \quad \frac{}{32}(0.05)^4 \quad 0.614 \quad 10^{\quad 6}\,\text{m}^4$$

Natural frequency of torsional vibration,

$$f_n \quad f_{nA} \text{ or } f_{nD}$$

$$\frac{1}{2}\sqrt{\frac{C.J}{l_A . I_A}} \quad \frac{1}{2}\sqrt{\frac{80 \quad 10^9 \quad 0.614 \quad 10^{\quad 6}}{0.52 \quad 216}} \text{ Hz} = 3.33 \text{ Hz} \quad \textbf{Ans.}$$

24.8. Free Torsional Vibrations of a Geared System

Consider a geared system as shown in Fig. 24.16 (*a*). It consists of a driving shaft *C* which carries a rotor *A*. It drives a driven shaft *D* which carries a rotor *B*, through a pinion *E* and a gear wheel *F*. This system may be replaced by an equivalent system of continuous shaft carrying a rotor *A* at one end and rotor *B* at the other end, as shown in Fig. 24.16 (*b*). It is assumed that

1. the gear teeth are rigid and are always in contact,

2. there is no backlash in the gearing, and

3. the inertia of the shafts and gears is negligible.

Let d_1 and d_2 = Diameter of the shafts *C* and *D*,

l_1 and l_2 = Length of the shafts *C* and *D*,

I_A and I_B = Mass moment of inertia of the rotors *A* and *B*,

$_A$ and $_B$ = Angular speed of the rotors *A* and *B*,

$$G \quad \text{Gear ratio} \quad \frac{\text{Speed of pinion } E}{\text{Speed of wheel } F} \quad \frac{_A}{_B}$$

. . . (Speeds of *E* and *F* will be same as that of rotors *A* and *B*)

d = Diameter of the equivalent shaft,

l = Length of the equivalent shaft, and

I_B = Mass moment of inertia of the equivalent rotor B .

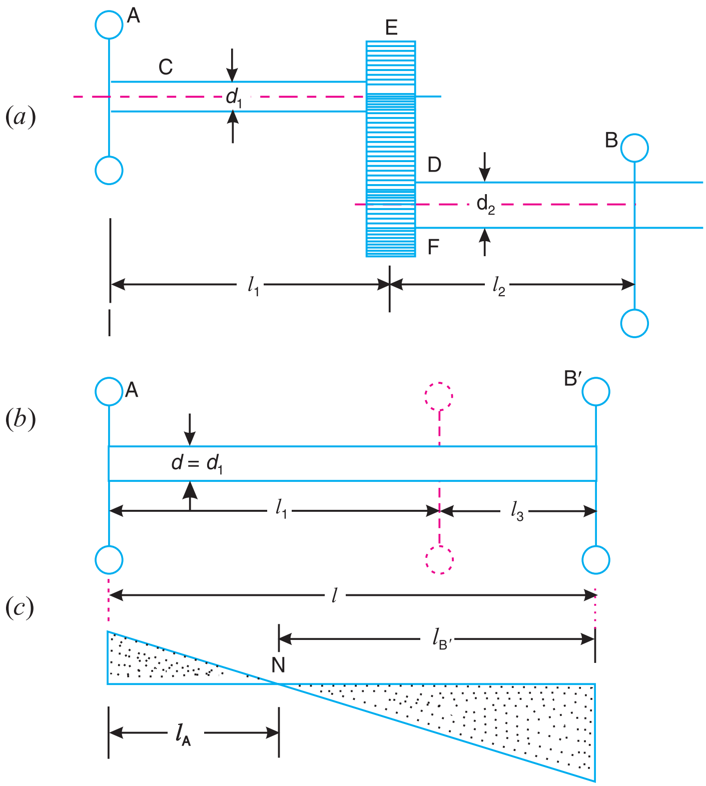


Fig. 24.16

The following two conditions must be satisfied by an equivalent system :

1. The kinetic energy of the equivalent system must be equal to the kinetic energy of the original system.

2. The strain energy of the equivalent system must be equal to the strain energy of the original system.

In order to satisfy the condition **(1)** for a given load,

K.E. of section l_1 + K.E. of section l_3

$$= \text{K.E. of section } l_1 + \text{K. E. of section } l_2$$

K.E. of section l_3 = K.E. of section l_2

or $$\frac{1}{2} \; I_B \, (\;\; _B)^2 \quad \frac{1}{2} \; I_B (\;\; _B)^2 \quad \text{or} \quad I_B \, (\;\; _A)^2 \quad I_B (\;\; _B)^2 \quad \ldots (\quad _B \quad _A)$$

$$I_B \quad I_B \left(\frac{_B}{_A}\right)^2 \quad \frac{I_B}{G^2} \qquad\qquad G \quad \frac{_A}{_B} \qquad \ldots \textit{(i)}$$

In order to satisfy the condition **(2)** for a given shaft diameter,

Strain energy of l_1 and l_2 = Strain energy of l_1 and l_2

Strain energy of l_3 = Strain energy of l_2

or $$\frac{1}{2} \; T_3 \quad _3 \quad \frac{1}{2} \; T_2 \quad _2 \quad \text{or} \quad \frac{T_3}{T_2} \quad \frac{_2}{_3} \qquad \ldots \textit{(ii)}$$

where T_2 and T_3 = Torque on the sections l_2 and l_3, and

$_2$ and $_3$ = Angle of twist on sections l_2 and l_3.

Assuming that the power transmitted in the sections l_3 and l_2 is same, therefore

$$T_3 \cdot {}_A \quad T_2 \cdot {}_B \quad \text{or} \quad \frac{T_3}{T_2} \quad \frac{_B}{_A} \quad \frac{1}{G} \qquad \ldots (iii)$$

Combining equations ***(ii)*** and ***(iii)***,

$$\frac{T_3}{T_2} \quad \frac{_2}{_3} \quad \frac{1}{G} \qquad \ldots (iv)$$

We know that torsional stiffness,

$$q \quad \frac{T}{} \quad \frac{C.J}{l}$$

where J = Polar moment of inertia of the shaft.

$$\text{For section } l_3, \quad \frac{T_3}{_3} \quad \frac{C.J_3}{l_3} \qquad \ldots (v)$$

and

$$\text{For section } l_2, \quad \frac{T_2}{_2} \quad \frac{C.J_2}{l_2} \qquad \ldots (vi)$$

Dividing equation ***(v)*** by equation ***(vi)***,

$$\frac{T_3}{T_2} \quad \frac{_2}{_3} \quad \frac{J_3}{l_3} \quad \frac{l_2}{J_2} \quad \text{or} \quad \frac{T_3}{T_2} \quad \frac{J_3 \cdot {}_3\, l_2}{J_2 \cdot {}_2 . l_3}$$

or

$$\frac{1}{G} \quad \frac{J_3}{J_2} \quad G \quad \frac{l_2}{l_3} \qquad \text{[From equation } (iv)]$$

$$l_3 \quad \frac{J_3}{J_2} \quad G^2 \quad l_2 \qquad \ldots (vii)$$

Assuming the diameter of the equivalent shaft as that of shaft C *i.e.* $d = d_1$, therefore

$$J_3 \quad \frac{}{32}(d_1)^4, \qquad \text{and} \qquad J_2 \quad \frac{}{32}(d_2)^4$$

$$\frac{J_3}{J_2} \quad \frac{d_1}{d_2}^{\;4}$$

Now the equation ***(vii)*** may be written as

$$l_3 \quad G^2 \quad l_2 \quad \frac{d_1}{d_2}^{\;4} \qquad \ldots (viii)$$

Thus the single shaft is equivalent to the original geared system, if the mass moment of inertia of the rotor B satisfies the equation ***(i)*** and the additional length of the equivalent shaft l_3 satisfies the equation ***(viii)***.

Length of the equivalent shaft,

$$l = l_1 + \; l_3 = l_1 + G^2.l_2 \quad \frac{d_1}{d_2}^{\;4} \qquad \ldots (ix)$$

Now, the natural frequency of the torsional vibration of a geared system (which have been reduced to two rotor system) may be determined as discussed below :

Let the node of the equivalent system lies at N as shown in Fig. 24.16 (c), then the natural frequency of torsional vibration of rotor A,

$$f_{nA} \quad \frac{1}{2}\sqrt{\frac{C.J}{l_A.I_A}}$$

and natural frequency of the torsional vibration of rotor B ,

$$f_{nB} \quad \frac{1}{2}\sqrt{\frac{C.J}{l_B\ .I_B}}$$

We know that $f_{nA} \quad f_{nB}$

$$\frac{1}{2}\sqrt{\frac{C.J}{l_A.I_A}} \quad \frac{1}{2}\sqrt{\frac{C.J}{l_B\ .I_B}}$$

or $$l_A.I_A \quad l_B\ .I_B \qquad . \qquad \ldots (x)$$

$$l_A \quad l_B \quad l \qquad \ldots (xi)$$

From these two equations (*x*) and (*xi*), the value of l_A and l_B may be obtained and hence the natural frequency of the torsional vibrations is evaluated.

Note : When the inertia of the gearing is taken into consideration, then an additional rotor [shown dotted in Fig. 24.16 (b)] must be introduced to the equivalent system at a distance l_1 from the rotor A. This rotor will have a mass moment of inertia $I_E \quad I_E \quad \frac{I_F}{G^2}$, where I_E and I_F are the moments of inertia of the pinion and wheel respectively. The system then becomes a three rotor system and the frequency of such a system may be obtained as discussed in the previous article.

Example 24.8. *A motor drives a centrifugal pump through gearing, the pump speed being one- third that of the motor. The shaft from the motor to the pinion is 60 mm diameter and 300 mm long. The moment of inertia of the motor is 400 kg-m^2. The impeller shaft is 100 mm diameter and 600 mm long. The moment of inertia of the impeller is 1500 kg-m^2. Neglecting inertia of the gears and the shaft, determine the frequency of torsional vibration of the system. The modulus of rigidity of the shaft material is 80 GN/m^2.*

Solution. Given : $G = N_A/N_B = 3$; $d_1 = 60$ mm $= 0.06$ m ; $l_1 = 300$ mm $= 0.3$ m ; $I_A = 400$ kg-m^2; $d^2 = 100$ mm $= 0.1$ m ; $l_2 = 600$ mm $= 0.6$ m ; $I_B = 1500$ kg-m^2 ; $C = 80$ GN/m$^2 = 80 \times 10^9$ N/m^2

The original and the equivalent system, neglecting the inertia of the gears, is shown in Fig. 24.17 (a) and (b) respectively. First of all, let us find the mass moment of inertia of the equivalent rotor B and the additional length of the equivalent shaft, assuming its diameter as $d_1 = 60$ mm.

We know that mass moment of the equivalent rotor B ,

$$I_B \quad I_B / G^2 \quad 1500/3^2 \quad 166.7 \ \text{kg–m}^2$$

and additional length of the equivalent shaft,

$$l_3 \quad G^2.l_2 \quad \frac{d_1}{d_2}^4 \quad 3^2 \quad 0.6 \quad \frac{0.06}{0.1}^4 \quad 0.7\text{m} \quad 700 \text{ mm}$$

Total length of the equivalent shaft,

$$l = l_1 + l_3 = 300 + 700 = 1000 \text{ mm} = 1 \text{ m}$$

Let the node of the equivalent system lies at N, as shown in Fig. 24.17 (c). We know that

$$l_A . I_A = l_B . I_B \quad \text{or} \quad l_A \times 400 = (1 - l_A)166.7 \quad \ldots (\because l_B = l - l_A)$$

$$\therefore \quad l_A = 0.294 \text{ m} = 294 \text{ mm}$$

We know that polar moment of inertia of the equivalent shaft,

$$J = \frac{\pi}{32}(d_1)^4 = \frac{\pi}{32}(0.06)^4 = 1.27 \times 10^{-6} \text{ m}^4$$

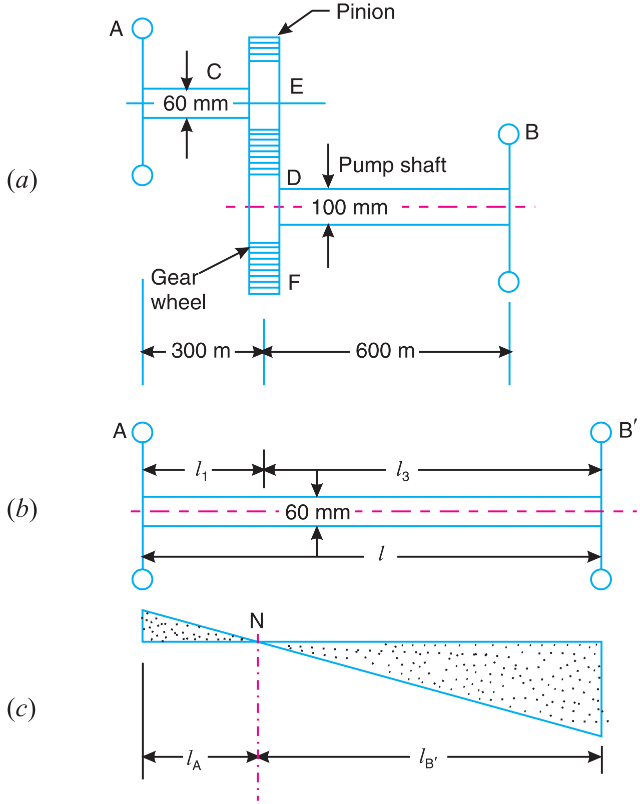


Fig. 24.17

Frequency of torsional vibration,

$$f_n = \frac{1}{2\pi}\sqrt{\frac{C.J}{l_A.I_A}} = \frac{1}{2\pi}\sqrt{\frac{80 \times 10^9 \times 1.27 \times 10^{-6}}{0.294 \times 400}} \text{ Hz}$$

$$= 4.7 \text{ Hz} \quad \textbf{Ans.}$$

Example 24.9. *An electric motor is to drive a centrifuge, running at four times the motor speed through a spur gear and pinion. The steel shaft from the motor to the gear wheel is 54 mm diameter and L metre long ; the shaft from the pinion to the centrifuge is 45 mm diameter and 400 mm long. The masses and radii of gyration of motor and centrifuge are respectively 37.5 kg, 100 mm ; 30 kg and 140 mm.*

Neglecting the inertia effect of the gears, find the value of L if the gears are to be at the node for torsional oscillation of the system and hence determine the frequency of torsional oscillation. Assume modulus of rigidity for material of shaft as 84 GN/m².

Solution. Given : $G = N_A/N_B = 1/4 = 0.25$; $d_1 = 54$ mm $= 0.054$ m ; $l_1 = L$ m ; $d_2 = 45$ mm $= 0.045$ m ; $l_2 = 400$ mm $= 0.4$ m ; $m_A = 37.5$ kg ; $k_A = 100$ mm $= 0.1$ m ; $m_B = 30$ kg ; $k_B = 140$ mm $= 0.14$ m; $C = 84$ GN/m² $= 84 \times 10^9$ N/m²

Value of L

We know that mass moment of inertia of the motor,

$$I_A \quad m_A (k_A)^2 \quad 37.5(0.1)^2 \quad 0.375 \text{ kg-m}^2$$

and mass moment of inertia of the centrifuge,

$$I_B \quad m_B (k_B)^2 \quad 30(0.14)^2 \quad 0.588 \text{ kg-m}^2$$

The original and the equivalent system, neglecting the inertia effect of the gears, is shown in Fig. 24.18 (*a*) and (*b*) respectively.

First of all, let us find the mass moment of inertia of the equivalent rotor B and the additional length of the equivalent shaft, keeping the diameter of the equivalent shaft as $d_1 = 54$ mm.

We know that mass moment of inertia of the equivalent rotor B ,

$$I_B \quad I_B / G^2 \quad 0.588/(0.25)^2 \quad 9.4 \text{ kg-m}^2$$

and additional length of equivalent shaft,

$$l_3 \quad G^2 . l_2 \quad \frac{d_1}{d_2}^4 \quad (0.25)^2 0.4 \quad \frac{0.054}{0.045}^4 \quad 0.0518 \text{ m}$$

Since the node N for torsional oscillation of the system lies at the gears, as shown in Fig. 24.18 (*c*), therefore

$$l_A \quad L, \quad \text{and} \quad l_B \quad l_3 \quad 0.0518 \text{ m}$$

We know that $\quad l_A . I_A \quad l_B . I_B$

$$L \times 0.375 = 0.0518 \times 9.4 = 0.487 \quad \text{or } L = 1.3 \text{ m } \textbf{Ans.}$$

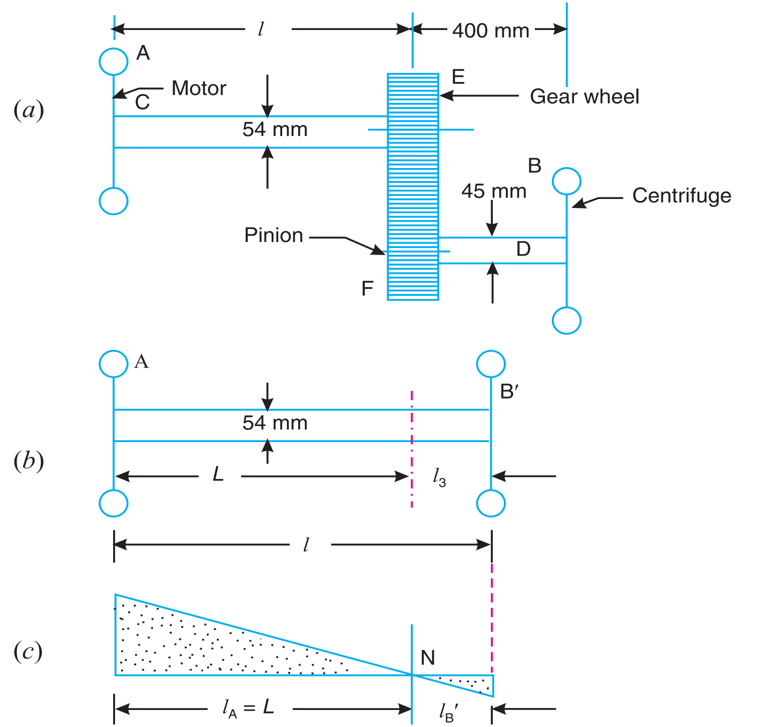


Fig. 24.18

Frequency of torsional oscillations

We know that polar moment of inertia of the equivalent shaft,

$$J \quad \frac{}{32}(d_1)^4 \quad \frac{}{32}(0.054)^4 \quad 0.835 \quad 10^{\ 6}\text{m}^4$$

Frequency of torsional oscillations,

$$f_n \quad \frac{1}{2}\sqrt{\frac{C.J}{l_A . I_A}} \quad \frac{1}{2}\sqrt{\frac{84 \quad 10^9 \quad 0.835 \quad 10^{\ 6}}{1.3 \quad 0.375}} = 60.4 \text{ Hz } \textbf{Ans.}$$