FINITE ELEMENT METHODS
Terminology

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1	**Finite Element Method**	1. A finite element method (abbreviated as FEM) is a **numerical** technique to obtain an **approximate** solution to a class of problems governed by **elliptic** partial differential equations. 2. The finite element method is a numerical technique. In this method all the complexities of the problems, like varying shape, boundary conditions and loads are maintained as they are but solutions obtained are approximate.
2	**FEM Packages**	NASTRAN, ANSYS, and ABAQUS , LS-DYNA, DEFORM etc.
3	**Stress**	When a material is subjected to an external force, a resisting force is set up within the component. The internal resistance force per unit area acting on a material or intensity of the forces distributed over a given section is called the stress at a point.
4	**Strain**	When a single force or a system force acts on a body, it undergoes some deformation. This deformation per unit length is known as strain. Mathematically strain may be defined as deformation per unit length.
5	**Tensile stress**	If $\boldsymbol{\sigma > 0}$ the stress is tensile. i.e. The fibres of the component tend to elongate due to the external force. A member subjected to an external force tensile P and tensile stress distribution due to the force.
6	**Compressive stress**	If $\boldsymbol{\sigma < 0}$ the stress is compressive. i.e. The fibres of the component tend to shorten due to the external force. A member subjected to an external compressive force P and compressive stress distribution due to the force.
7	**Shear stress**	When forces are transmitted from one part of a body to other, the stresses developed in a plane parallel to the applied force are the shear stress. Shear stress acts parallel to plane of interest.
8	**Shear Strain**	The distortion produced by shear stress on an element or rectangular block is shown in the figure. The shear strain or 'slide' is expressed by angle ϕ and it can be defined as the change in the right angle. It is measured in radians and is dimensionless in nature.
9	**Poisson's Ratio**	The ratio lateral strain to longitudinal strain produced by a single stress is known as Poisson's ratio. Symbol used for poisson's ratio is $1/m$.
10	**Lateral Strain**	**Lateral strain,** also known as transverse strain, is defined as the ratio of the change in diameter of a circular bar of a material due to deformation in the longitudinal direction.

11	**Elasticity**	This is the property of a material to regain its original shape after deformation when the external forces are removed.
12	**Plasticity**	When the stress in the material exceeds the elastic limit, the material enters into plastic phase where the strain can no longer be completely removed. Under plastic conditions materials ideally deform without any increase in stress.
13	**Modulus of Rigidity(G)**	For elastic materials it is found that shear stress is proportional to the shear strain within elastic limit. The ratio is called modulus rigidity. It is denoted by the symbol 'G' or 'C'.
14	**Bulk modulus (K)**	It is defined as the ratio of uniform stress intensity to the volumetric strain. It is denoted by the symbol K.
15	**Body force**	It is defined as distribution force per unit volume
16	**Traction force**	It is defined as force per unit area
17	**Stress And Equilibrium Equations**	$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} + f_x = 0$ $\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} + f_y = 0$ $\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} + f_z = 0$
18	**Strain – Displacement Relations**	$\varepsilon = \left[\frac{\partial u}{\partial x} \quad \frac{\partial v}{\partial y} \quad \frac{\partial w}{\partial z} \quad \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \quad \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \quad \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right]^T$
19	**Stress – Strain Relations**	$\epsilon_x = \frac{\sigma_x}{E} - \nu\frac{\sigma_y}{E} - \nu\frac{\sigma_z}{E}$, $\epsilon_y = -\nu\frac{\sigma_x}{E} + \frac{\sigma_y}{E} - \nu\frac{\sigma_z}{E}$, $\epsilon_z = \mathbf{v}\frac{\sigma_x}{E} - \nu\frac{\sigma_y}{E} + \frac{\sigma_z}{E}$
20	**Plane Stress Conditions**	Plane stress is defined to be a state of stress in which the normal stress and the shear stresses directed Perpendicular to the plane are assumed to be zero. That is, the normal stress $\sigma_z$and the shear stresses τ_{xz} and τ_{yz} are assumed to be zero.
21	**Plane Strain Conditions**	Plane strain is defined to be a state of strain in which the strain normal to the x-y plane $\epsilon_z$and the shear strains $\gamma_{xz}$and $\gamma_{yz}$are assumed to be zero.
22	**Variational Methods**	In variational technique, the calculus of variation is used to obtain the integral form corresponding to the given differential equation. This integral needs to be minimized to obtain the solution of the problem. For structural mechanics problems, the integral form turns out to be the expression for the total potential energy of the structure.

23	**Weighted Residual Methods**	In weighted residual technique, the integral form is constructed as a weighted integral of the governing differential equation where the weight functions are known and arbitrary except that they satisfy certain boundary conditions. To reduce the continuity requirement of the solution, this integral form is often modified using the divergence theorem. This integral form is set to zero to obtain the solution of the problem. For structural mechanics problems, if the weight function is considered as the virtual displacement, then the integral form becomes the expression of the virtual work of the structure.
24	**Total Potential Energy**	The total potential energy of an elastic body , is defined as the sum of total strain energy (U) and the work potential (WP) . $\Pi = U + WP$
25	**Principle Of Minimum Potential Energy**	For conservative systems, of all the kinematically admissible displacement fields, those corresponding to equilibrium extremize the total potential energy. If the extremum condition is a minimum, the equilibrium state is stable.
26	**Discretization of Domain**	The process of Dividing the domain into discrete elements is called discretization
27	**Interpolation functions**	The **shape function** or interpolate function is the **function** which interpolates the solution between the discrete values obtained at the mesh nodes
28	**Convergence Requirements**	1.Completeness 2.compatibility
29	**Boundary conditions**	The values of variables prescribed on the boundaries of the region are called as boundary conditions.
30	**Geometric (Essential or Dirichlet) Boundary conditions**	which are imposed on the primary variable like displacements
31	**Force (Natural) Boundary conditions**	**Natural** or force **boundary conditions** which are imposed on the secondary variable like forces and tractions.
32	**Stiffness Matrix**	stiffness matrix contains the geometric and material behaviour information that indicates the resistance of the element to deformation when subjected to loading.
33	**Global coordinate system**	The coordinate system used to define the points in the entire Structure is called “Global coordinates system.” Generally Cartesian coordinates system is used as “Global coordinates system.”

34	**Local coordinate system**	For the convenience of deriving element properties For each element a separate coordinate system is used, called “Local coordinate system.”
35	**Constant Strain Triangle**	If the field variables such as strains and heat flux will be linear and constant throughout an element then it is called “Constant Strain Triangle.”
36	**Higher order element**	if the interpolation polynomial is of order two or more, the element is known as a higher order element.
37	**Isoparametric Element**	In an element, if the number of nodes used for defining the Geometry is same as the number of nodes used for defining the displacement, then it is called “Isoparametric Element.”
38	**Superparametric Element**	In an element, if the number of nodes used for defining the Geometry is more as the number of nodes used for defining the displacement, then it is called “Superparametric Element.”
39	**Lumped mass matrix**	Total mass of the element is assumed equally distributed at all the nodes of the element in each of the translational degrees of freedom. Lumped mass is not used for rotational degrees of freedom. Off-diagonal elements of this matrix are all zero.
40	**Consistent mass matrix**	1.The mass of each element is equally distributed at all the nodes of that Element 2. Mass, being a scalar quantity, has same effect along the three translational degrees of freedom (u, v and w) and is not shared 3. Mass, being a scalar quantity, is not influenced by the local or global coordinate system. Hence, no transformation matrix is used for converting mass matrix from element (or local) coordinate system to structural (or global) coordinate system.

Introduction
to
Finite Element Method

Learning Objectives

At the end of this topic, you will be able to:

- ❖ Know the finite element methods and its applications
- ❖ Derive stress and equilibrium relations
- ❖ Explain strain displacement and stress strain relations
- ❖ Understand the concepts of plane stress and plane strain conditions

Outcomes

By the end of this topic, you will be able to:

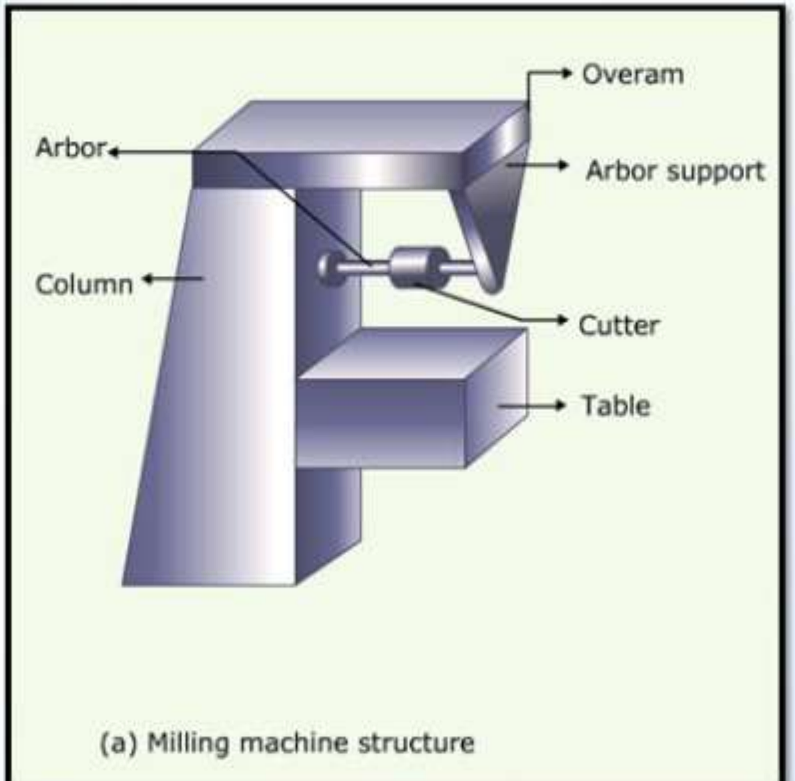
- ❖ Discuss the finite element methods and its applications
- ❖ Illustrate stress and equilibrium relations
- ❖ Understand strain displacement and stress strain relations
- ❖ Explain the concepts of plane stress and plane strain conditions

Finite Element Methods

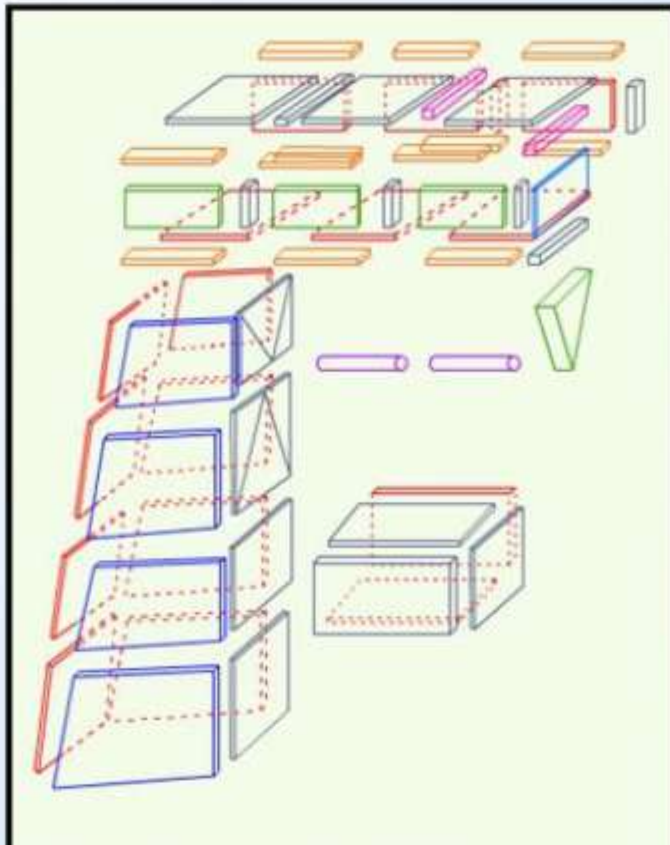
- The finite element method has become a powerful tool for the numerical solution of a wide range of engineering problems
- With the advances in computer technology and CAD systems, complex problems can be modeled with relative ease. Several alternative configurations can be tested on a computer before the first prototype is built
- In this method of analysis, a complex region defining a continuum is discretized into simple geometric shapes called finite elements.
- The basic idea in the finite element method is to find the solution of a complicated problem by replacing it by a simpler one. Since the actual problem is replaced by a simpler one in finding the solution, we will be able to find only an approximate solution rather than the exact solution.
- The material properties and· the governing relationships are considered over these elements and expressed in terms of unknown values at element corners.

Example of Finite Element Method

- An assembly process, duly considering the loading and constraints, results in a set of equations. Solution of these equations gives us the approximate behavior of the continuum.



(a) Milling machine structure



structures to walls, bridges, and pre stressed stability of structures periodic loads concrete structures

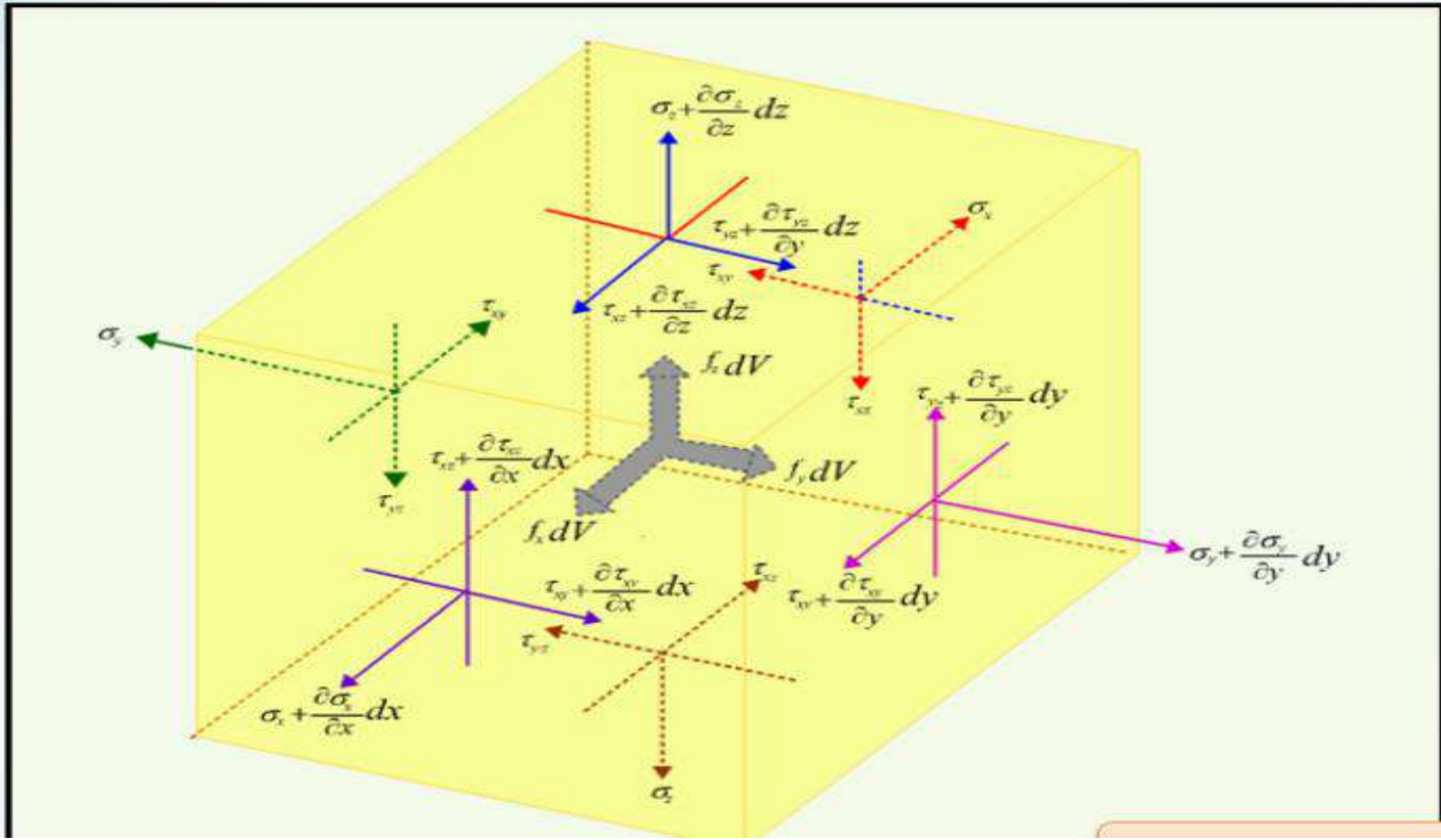
Aircraft Structures:

- Static analysis of aircraft wings, Natural frequencies, Response of aircraft structures to fuselages, fins, rockets, flutter, and stability of random loads; dynamic response spacecraft, and missile structures aircraft, rocket, of aircraft and spacecraft to spacecraft, and missile periodic loads structures

Heat Conduction:

- Steady-state temperature Transient heat flow in rocket distribution in solids and fluids nozzles, internal combustion engines, turbine blades, fins, and building structures

Stress and Equilibrium



Stress and Equilibrium

- The deformation of a point $x\,(=[x,y,z]^T)$ is given by the three components of its displacement

$$u=[u,v,w]^T \longrightarrow 1$$

- The distributed force per unit volume, for example, the weight per unit volume, is the vector f given by,

$$f=[f_x,f_y,f_z]^T \longrightarrow 2$$

- The body force acting on the elemental volume dV , The surface traction T may be given by its component values at points on the surface:

$$\mathrm{T}=\left[f_x,f_y,f_z\right]^T \longrightarrow 3$$

- Examples of traction are distributed contact force and action of pressure. A load P acting at a point i is represented by its three components:

Stress and Equilibrium

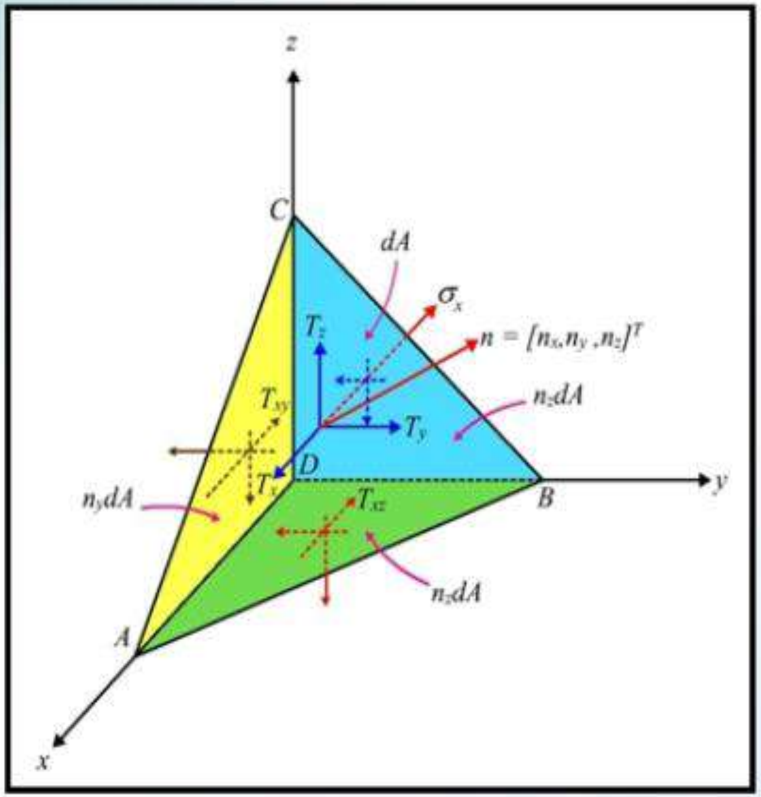
- However, we represent stress by the six independent components as in,

$$\sigma = \sigma_x, \sigma_y, \sigma_z, T_{yz}, T_{xz}, T_{xy} \longrightarrow 5$$

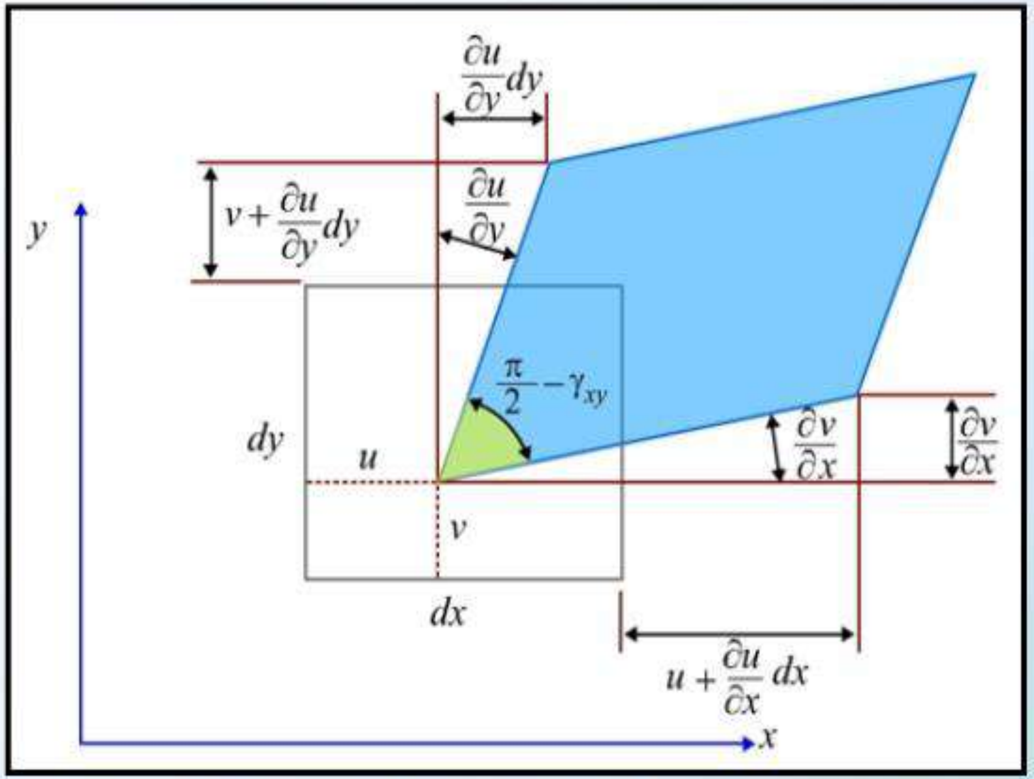
- First we get forces on faces by multiplying the stresses by the corresponding areas. Writing all the forces in X, Y and Z directions are equated to Zero *and recognizing* $dV = dx$, dy, dz, *we get the equilibrium equations.*

$$\begin{aligned}
\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} + f_x = 0 \\
\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} + f_y = 0 \\
\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} + f_z = 0
\end{aligned} \longrightarrow 6$$

Strain Displacement Relations



An elemental volume at surface(3D)



Deformed elemental surface

Strain Displacement Relations

The Normal strain on the X, Y and Z directions can be obtained as follows,

$$\in_x = \frac{\sigma_x}{E} - v\frac{\sigma_y}{E} - v\frac{\sigma_z}{E}$$

$$\in_x = -v\frac{\sigma_x}{E} + \frac{\sigma_y}{E} - v\frac{\sigma_z}{E} \longrightarrow 7$$

$$\in_x = -v\frac{\sigma_x}{E} - v\frac{\sigma_y}{E} + \frac{\sigma_z}{E}$$

The Shear strain on the 3 directions will be ,

$$\gamma_{yz} = \frac{\tau_{yz}}{G}$$

$$\gamma_{xz} = \frac{\tau_{xz}}{G} \longrightarrow 8$$

$$\gamma_{xy} = \frac{\tau_{xy}}{G}$$

The strains in all three directions can be expressed as, when the equation 6 and 7 can be substituted in the equation 8 can get a matrix form . The matrix form has to be transported

$$\in = \{\in_x, \in_y, \in_z, \gamma_{yz}, \gamma_{xz}, \gamma_{xy}\}^T \longrightarrow 9$$

Strain Displacement Relations

The deformation of the dx - dy face for small deformations, which we consider here. Finally the strain displacement matrix has been obtained.

$$\in=\left[\frac{\partial u}{\partial x},\frac{\partial v}{\partial y},\frac{\partial w}{\partial z},\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y},\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x},\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right]^T \longrightarrow 10$$

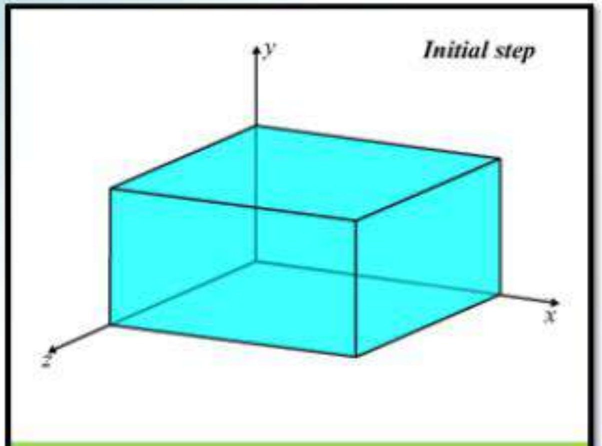
For linear elastic materials, the stress-strain relations come from the generalized Hooke's law. For isotropic materials, the two material properties are Young's modulus (or modulus of elasticity) *and Poisson's ratio Considering an elemental cube inside the* body, Hooke's law gives

$$\in_x=\frac{\sigma_x}{E}-v\frac{\sigma_y}{E}-v\frac{\sigma_z}{E} \qquad \gamma_{yz}=\frac{T_{yz}}{G}$$

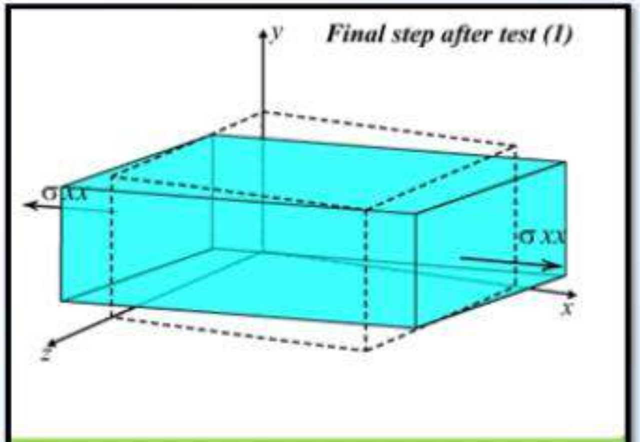
$$\in_x=-v\frac{\sigma_x}{E}+\frac{\sigma_y}{E}-v\frac{\sigma_z}{E} \qquad \gamma_{xz}=\frac{T_{xz}}{G}$$

$$\in_x=-v\frac{\sigma_x}{E}-v\frac{\sigma_y}{E}+\frac{\sigma_z}{E} \qquad \gamma_{xy}=\frac{T_{xy}}{G}$$

Strain Displacement Relations



(a) Initial Shape of cube



b) Final shape (Elongation in x direction)

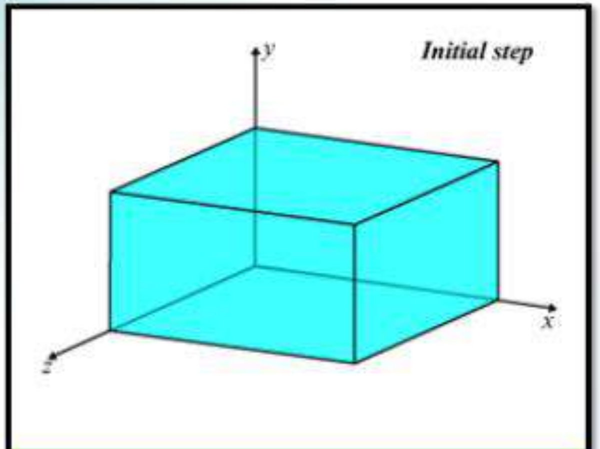
In the first test elongation has been encountered in X direction so that the length has been increased while breath/height decreases

$$\epsilon_{xx}^{(1)} = \frac{\sigma_{xx}}{E}$$

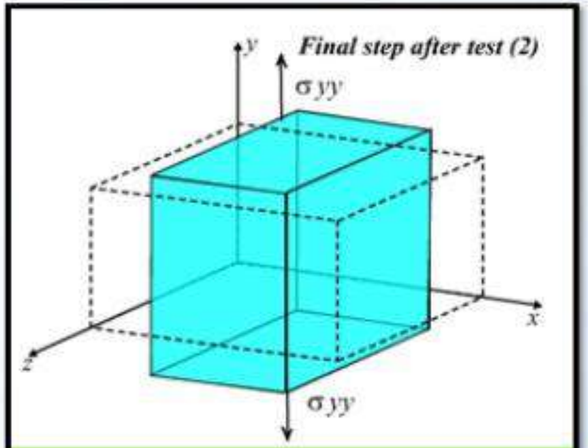
$$\epsilon_{yy}^{(1)} = -\frac{v\sigma_{xx}}{E}$$

$$\epsilon_{zz}^{(1)} = -\frac{v\sigma_{xx}}{E}$$

Strain Displacement Relations



(a) Initial Shape of cube



c) Final shape (Elongation in y direction)

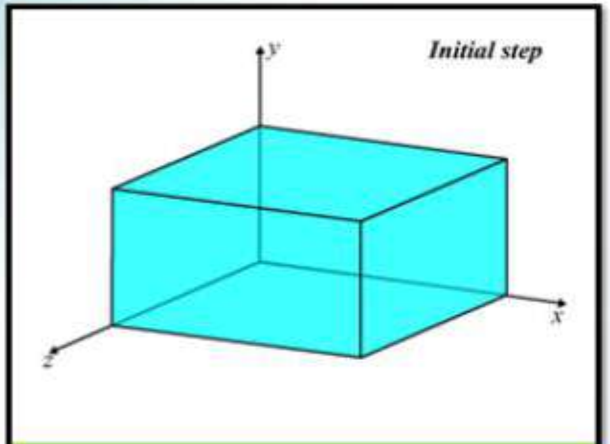
In the second test elongation has been encountered in Y direction so that the length has been decreased while breath/height increases

$$\in_{xx}^{(2)} = -\frac{v\,\sigma_{zz}}{E}$$

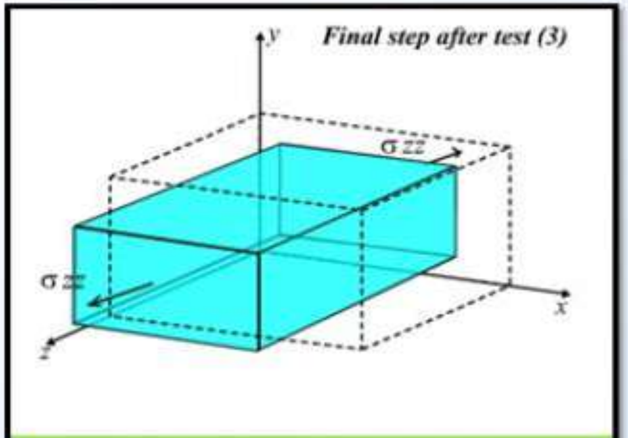
$$\in_{yy}^{(2)} = -\frac{v\,\sigma_{zz}}{E}$$

$$\in_{zz}^{(2)} = \frac{\sigma_{yy}}{E}$$

Strain Displacement Relations



(a) Initial Shape of cube



d) Final shape (Elongation in z direction)

In the Third test elongation has been encountered in Z direction so that the length is constant while span increases

$$\epsilon_{xx}^{(3)} = -\frac{v\sigma_{zz}}{E}$$

$$\epsilon_{yy}^{(3)} = -\frac{v\sigma_{zz}}{E}$$

$$\epsilon_{zz}^{(3)} = \frac{\sigma_{yy}}{E}$$

Stress Strain Relations

In the general case the cube is subjected to combined normal stresses σ_{xx}, σ_{yy} and σ_{zz}. Since we assumed that the material is linearly elastic, the combined strains can be obtained by superposition of the foregoing results:

$$\in_{xx}=\in_{xx}^{(1)}+\in_{xx}^{(2)}+\in_{xx}^{(3)}=\frac{\sigma_{xx}}{E}-\frac{v\sigma_{yy}}{E}-\frac{v\sigma_{zz}}{E}=\frac{1}{E}(\sigma_{xx}-v\sigma_{yy}-v\sigma_{zz})$$

$$\in_{yy}=\in_{yy}^{(1)}+\in_{yy}^{(2)}+\in_{yy}^{(3)}=-\frac{v\sigma_{xx}}{E}+\frac{\sigma_{yy}}{E}-\frac{v\sigma_{zz}}{E}=\frac{1}{E}(-v\sigma_{xx}+\sigma_{yy}-v\sigma_{zz})$$

$$\in_{zz}=\in_{zz}^{(1)}+\in_{zz}^{(2)}+\in_{zz}^{(3)}=-\frac{v\sigma_{xx}}{E}-\frac{v\sigma_{yy}}{E}+\frac{\sigma_{zz}}{E}=\frac{1}{E}(-v\sigma_{xx}-v\sigma_{yy}+\sigma_{zz})$$

$$\gamma_{xy}=\gamma_{yx}=\frac{\tau_{xy}}{G}=\frac{\tau_{yx}}{G},\gamma_{yz}=\gamma_{zy}=\frac{\tau_{yz}}{G}=\frac{\tau_{zy}}{G},\gamma_{zx}=\gamma_{xz}=\frac{\tau_{zx}}{G}=\frac{\tau_{xz}}{G}$$

Stress Strain Relations

After substituting all the stress and strain values in matrix form will get the stress strain equations

$$\begin{bmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \epsilon_{xx} \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{bmatrix} = \begin{bmatrix} \frac{1}{E} & -\frac{v}{E} & -\frac{v}{E} & 0 & 0 & 0 \\ -\frac{v}{E} & \frac{1}{E} & -\frac{v}{E} & 0 & 0 & 0 \\ -\frac{v}{E} & -\frac{v}{E} & \frac{1}{E} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{G} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{G} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{G} \end{bmatrix} \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{xz} \end{bmatrix}$$

Plane Stress Condition

- Two specializations of the foregoing 3D equations to two dimensions are of interest in the applications: plane stress and plane strain. Plane stress is more important in Aerospace structures, which tend to be thin.

- In this case all stress components with a z component are assumed to vanish. For a linearly elastic isotropic material, the strain and stress matrices take on the form. There are two matrices which shows the stresses and strains in the element while plane stress condition exist

$$\begin{bmatrix} \epsilon_{xx} & \gamma_{xy} & 0 \\ \gamma_{yx} & \epsilon_{yy} & 0 \\ 0 & 0 & \epsilon_{zz} \end{bmatrix} \quad \begin{bmatrix} \sigma_{xx} & \tau_{xy} & 0 \\ \tau_{yx} & \sigma_{yy} & 0 \\ 0 & 0 & \sigma_{zz} \end{bmatrix}$$

Plane Stress Condition

❖ When substitute the values of the stress and strain matrix the plane stress condition exist as follows,

$$\in_{xx} = \frac{1}{E}(\sigma_{xx} - v\sigma_{yy})$$

$$\in_{yy} = \frac{1}{E}(-v\sigma_{xx} - v\sigma_{yy})$$

$$\in_{zz} = \frac{1}{E}(\sigma_{xx} + \sigma_{yy})$$

$$\gamma_{xy} = \frac{\tau_{xy}}{G}$$

$$\gamma_{yz} = \gamma_{zx} = 0$$

Plane Stress Condition

- The final expression for the stress strain equation in plane stress condition is,

$$\begin{bmatrix} \in_{xx} \\ \in_{yy} \\ \in_{zz} \\ \gamma_{xy} \end{bmatrix} = \begin{bmatrix} \frac{1}{E} & -\frac{v}{E} & 0 \\ -\frac{v}{E} & \frac{1}{E} & 0 \\ -\frac{v}{E} & -\frac{v}{E} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{bmatrix}$$

- Inverting the matrix composed by the first, second and fourth rows of the above relation gives the stress-strain equations

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{bmatrix} = \begin{bmatrix} E & Ev & 0 \\ Ev & E & 0 \\ 0 & 0 & G \end{bmatrix} \begin{bmatrix} \in_{xx} \\ \in_{yy} \\ \gamma_{xy} \end{bmatrix}$$

Plane Stress Condition

- The final expression for the stress strain equation in plane stress condition is,

$$\begin{bmatrix} \in_{xx} \\ \in_{yy} \\ \in_{zz} \\ \gamma_{xy} \end{bmatrix} = \begin{bmatrix} \frac{1}{E} & -\frac{v}{E} & 0 \\ -\frac{v}{E} & \frac{1}{E} & 0 \\ -\frac{v}{E} & -\frac{v}{E} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{bmatrix}$$

- Inverting the matrix composed by the first, second and fourth rows of the above relation gives the stress-strain equations

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{bmatrix} = \begin{bmatrix} E & Ev & 0 \\ Ev & E & 0 \\ 0 & 0 & G \end{bmatrix} \begin{bmatrix} \in_{xx} \\ \in_{yy} \\ \gamma_{xy} \end{bmatrix}$$

Plane Stress Condition(Assumptions)

- In this case all strain components with a z component are assumed to vanish. For a linearly elastic isotropic material, the strain and stress matrices take on the form.
- The normal strain in the Z direction is said to be Zero for this condition and also the shear strain in the xz , yz ,zz and zy tends to zero
- In the same way the normal stress in all directions have some finite values. The shear stress in the xz, yz, zx and zy tends to zero

$$\begin{bmatrix} \in_{xx} & \gamma_{xy} & 0 \\ \gamma_{yx} & \in_{yy} & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} \sigma_{xx} & \tau_{xy} & 0 \\ \tau_{yx} & \sigma_{yy} & 0 \\ 0 & 0 & \sigma_{zz} \end{bmatrix}$$

Plane Stress Condition(Assumptions)

- After substituting the normal and shear stresses and strains the final equations are attained as follows.

$$\sigma_{xx} = \frac{E}{(1-2v)(1+v)}[(1-v)\in_{xx} + v\in_{yy}]$$

$$\sigma_{yy} = \frac{E}{(1-2v)(1+v)}[v\in_{xx} + (1-v)\in_{yy}]$$

$$\sigma_{zz} = \frac{E}{(1-2v)(1+v)}[v\in_{xx} + v\in_{yy}]$$

$$\tau_{xy} = G_{\gamma xy},\ \tau_{yz} = 0,\ \tau_{zx} = 0$$

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{bmatrix} = \begin{bmatrix} \hat{E}(1-v) & \hat{E}v & 0 \\ \hat{E}v & \hat{E}(1-v) & 0 \\ \hat{E}v & \hat{E}v & 0 \\ 0 & 0 & G \end{bmatrix} \begin{bmatrix} \in_{xx} \\ \in_{yy} \\ \in_{zz} \\ \gamma_{xy} \end{bmatrix}$$

Concept of Variational Principles

Learning Objectives

At the end of this topic, you will be able to:

- Explain the concept of variational principles
- Explain weighted residual method principles
- Know concept of potential energy
- Understand one dimensional problems

Outcomes

By the end of this topic, you will be able to:

- Understand the concept of variational principles
- Discuss about weighted residual method principles
- Understand the concept of potential energy
- Solve one dimensional problems

Variational Method (Rayleigh Ritz)/Concept of Potential Energy

- In mechanics of solids, our problem is to determine the displacement of the body satisfying the equilibrium equations.
- Normally stresses are related to strains, which, in turn, are related to displacements.
- This leads to requiring solution of second order partial differential equations. Solution of this set of equations is generally referred to as an exact solution.
- Such exact solutions are available for simple geometries and loading conditions, and one may refer to publications in theory of elasticity.
- For problems of complex geometries and general boundary and loading conditions. obtaining such solutions is an almost impossible task.
- Approximate solution methods usually employ potential energy or variational methods, which place less stringent conditions on the functions.

Variational Method (Rayleigh Ritz)/Concept of Potential Energy

Principle of Minimum Potential Energy

For conservative systems, of all the kinematically admissible displacement fields, those corresponding to equilibrium extremize the total potential energy. If the extremum condition is a minimum, the equilibrium state is stable.

The total potential energy of an elastic body, is defined as the sum of total strain energy and the work potential: Potential Energy = Strain energy + Work potential

The total strain energy U is given by,

$$U = \frac{1}{2}\int_v \sigma^T \varepsilon dv \longrightarrow 1$$

The work potential WP is given by

$$WP = -\int_v \sigma^T f dv - \int_s u^T T ds - \sum_i u_i^T P_i \longrightarrow 2$$

The total potential for the general elastic body

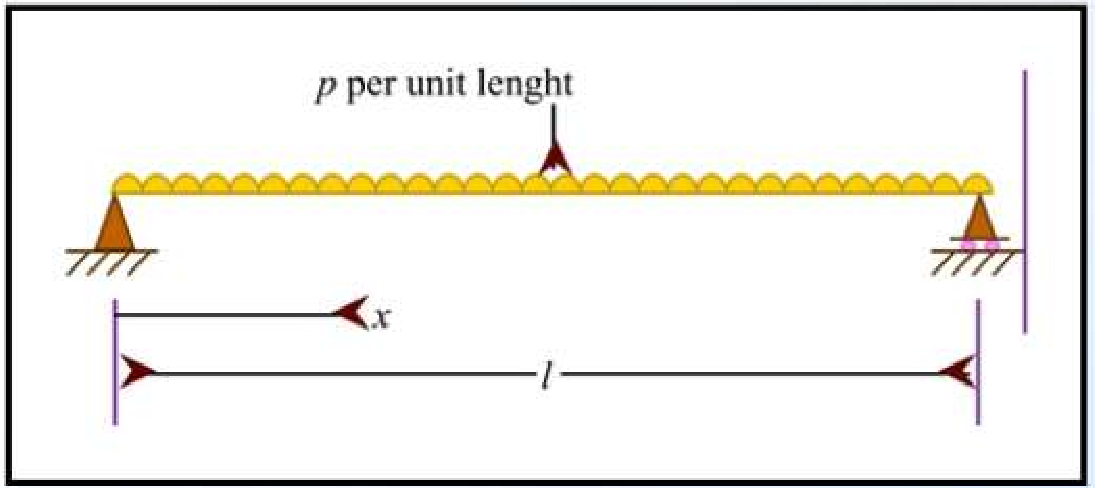
$$\pi = \frac{1}{2}\int_v \sigma^T \varepsilon dv - \int_v u^T f dV - \int_s u^T T dS \sum_i u_i^T P_i \longrightarrow 3$$

Advantages of Variational Method (Rayleigh Ritz)

- The functional I contains lower order derivatives of the field variable compared to the governing differential equation and hence an approximate solution can be obtained using a larger class of functions
- Sometimes the problem may possess a dual variation formulation, in which case the solution can be sought either by minimizing (or maximizing) the functional I or by maximizing (or minimizing) its dual functional.
- In such cases, one can find an upper and a lower bound to the solution and estimate the order of error in either of the approximate solutions obtained.
- Using variational formulation, it is possible to prove the existence of solution in some cases.
- The variational formulation permits us to treat complicated boundary conditions as natural or free boundary conditions. Thus, we need to explicitly impose only the geometric or forced boundary conditions in the finite element method, and the variational statement implicitly imposes the natural boundary conditions.

Problems in Variational Approach

- Find the approximate deflection of a simply supported beam under a uniformly distributed load p using the Rayleigh-Ritz method



Approach; Find the functional Whose extermination yields the differential equation governing the deflection of the beam. Assume an approximate solution satisfying the boundary conditions in terms of two unknown constants and evaluate the constants using the conditions of extermination of 1.

Problems in Variational Approach

Solution: Let w(x) denote the deflection of the beam (field variable). The differential equation formulation leads to the following statement of the problem. The governing equation is,

$$EI\frac{d^4w}{dx^4}-p=0;\quad 0\le x\le 1 \qquad \text{--------------(4)}$$

The boundary conditions are,

$$\left.\begin{aligned} &w(x=0)=w(x=1)=0\ \textit{(deflection zero at ends)}\\ &EI\frac{d^2w}{dx^2}(x=0)=EI\frac{d^2w}{dx^2}(x=1)=0\ \textit{(bending moment zero at ends)}\end{aligned}\right\} \qquad \text{--------------(5)}$$

To Find w(x) that minimizes the integral,

$$A=\int_{x=0}^{1}F.dx=\frac{1}{2}\int_0^1\left[EI\left(\frac{d^2w}{dx^2}\right)^2-2p.w\right]dx \qquad \text{--------------(6)}$$

w(x) by orthogonal functions of the sinusoidal type.

$$\underset{\sim}{w}(x)=C_1\sin\left(\frac{\pi x}{l}\right)+C_2\sin\left(\frac{3\pi x}{l}\right)=C_1f_1(x)+C_2f_2(x) \qquad \text{--------------(7)}$$

Problems in Variational Approach

$$A=\int_0^1\left[\frac{EI}{2}\left\{C_1\left(\frac{\pi}{l}\right)^2\sin\left(\frac{\pi x}{l}\right)+C_2\left(\frac{3\pi}{l}\right)^2\sin\left(\frac{3\pi x}{l}\right)\right\}^2-p\left\{C_1\sin\left(\frac{\pi x}{l}\right)+C_2\sin\left(\frac{3\pi x}{l}\right)\right\}\right]dx$$

$$=\int_0^1\left[\frac{EI}{2}\left\{C_1^2\left(\frac{\pi}{l}\right)^4\sin^2\left(\frac{\pi x}{l}\right)+C_2^2\left(\frac{3\pi}{l}\right)^4\sin\left(\frac{3\pi x}{l}\right)+2C_1C_2\left(\frac{\pi}{l}\right)^2\left(\frac{3\pi}{l}\right)^2\right.\right.$$

$$\left.\left.\cdot\sin\left(\frac{\pi x}{l}\right)\sin\left(\frac{3\pi x}{l}\right)\right\}-p\left\{C_1\sin\left(\frac{\pi x}{l}\right)+C_2\sin\left(\frac{3\pi x}{l}\right)\right\}\right]dx$$

By using the relations,

$$\int_0^1\sin\left(\frac{m\pi x}{l}\right)\sin\left(\frac{n\pi x}{l}\right)dx=\begin{cases}0 & if\ m\neq n\\ 1/2 & if\ m\neq n\end{cases}$$

$$\int_0^1\sin\left(\frac{m\pi x}{l}\right)dx=\frac{2l}{m\pi}\quad if\ m\ is\ an\ odd\ \text{integer}$$

$$A=\frac{EI}{2}\left\{C_1^2\left(\frac{\pi}{l}\right)^4\frac{l}{2}+C_2^2\left(\frac{3\pi}{l}\right)^4\frac{l}{2}\right\}-p\left\{C_1\frac{2l}{\pi}+C_2\frac{2l}{3\pi}\right\}$$

Problems in Variational Approach

where C, and C2 are independent constants. For the minimum of A, we have,

$$\frac{\partial A}{\partial C_1} = \frac{EI}{2}\left\{2C_1\left(\frac{\pi}{l}\right)^4 \frac{l}{2}\right\} - p\frac{2l}{\pi} = 0$$

$$\frac{\partial A}{\partial C_2} = \frac{EI}{2}\left\{2C_2\left(\frac{3\pi}{l}\right)^4 \frac{l}{2}\right\} - p\frac{2l}{3\pi} = 0$$

The values of the constants,

$$C_1 = \frac{4pl^4}{\pi^5 EI} \quad and \quad C_2 = \frac{4pl^4}{243\pi^5 EI}$$

The values obtained for the full length is,

$$\underset{\sim}{\psi}(x) = \frac{4pl^4}{\pi^5 EI}\left[\sin\left(\frac{\pi x}{l}\right) + \frac{1}{243}\sin\left(\frac{3\pi x}{l}\right)\right]$$

The values obtained for the Mid length is,

$$\underset{\sim}{\psi}(x = l/2) = \frac{968}{243\pi^5}\frac{pl^4}{EI} = \frac{1}{76.5}\frac{pl^4}{EI}$$

Weighted Residual Method

Methods of weighted residuals

- Point collocation method
- Sub domain collocation method
- Galerkin Method
- Least square method

Collocation (or Point Collocation) Method

In this method, the residual R is set equal to zero at n points in the domain V, thereby implying that the parameters C_j are to be selected such that the trial function $I/I(x)$ represents $I/I(x)$ at these n points exactly. This procedure yields n simultaneous-algebraic equations in the unknowns C_j $(i = 1, 2, \ldots, n)$. The collocation points X_i at which, $j = 1, 2, \ldots, n$ are usually chosen to cover the domain V more or less uniformly in some simple patter

$$f(R) = R \text{ and } w = \delta(x_j - x) \longrightarrow 4$$

where δ indicates the Dirac delta function, X_j denotes the position of the j-th point, and x gives the position of a general point in the domain V. Thus, $W = 1$ at point $x = X_i$ and zero elsewhere in the domain $V (j = 1, 2, \ldots, n)$.

Problems on Collocation (or Point Collocation) Method

- Find the solution of the differential equation $\dfrac{d^2\phi}{dx^2}+\phi+x=0;\quad 0\le x\le 1$. Subjected to the boundary conditions $\phi(0)=\phi(1)$ using a point collocation method, find the solution of the differential equation.

Approach: Assume an approximate solution satisfying the boundary conditions with two unknown constants. Set the residue equal to zero at the collocation points to evaluate the constants

Solution:

The approximate solution satisfying the boundary conditions is taken as,

$$\bar{\phi}(x)=c_1x(1-x)+c_2x^2(1-x)$$

Problems on Collocation (or Point Collocation) Method

Where, C1 and C2 are unknown constants. Using this solution, the residue can be expressed as,

$$R = \frac{d^2\bar{\phi}}{dx^2} + \bar{\phi} + x = c_1(-2 + x - x^2) + c_2(2 - 6x + x^2 - x^3) + x$$

The residue is set equal to zero at each of the collocation points:

$$R\left(x = \frac{1}{4}\right) = c_1\left(-2 + \frac{1}{4} - \frac{1}{16}\right) + c_2\left(2 - \frac{3}{2} + \frac{1}{16} - \frac{1}{64}\right) + \frac{1}{4}$$

$$= -\frac{29}{16}c_1 + \frac{35}{64}c_2 + \frac{1}{4} = 0$$

$$R\left(x = \frac{1}{2}\right) = c_1\left(-2 + \frac{1}{2} - \frac{1}{4}\right) + c_2\left(2 - 3 + \frac{1}{4} - \frac{1}{8}\right) + \frac{1}{2}$$

$$= -\frac{7}{4}c_1 + \frac{7}{8}c_2 + \frac{1}{2} = 0$$

Sub Domain Collocation Method

- Here, the domain V is first subdivided into n subdomains, $i = 1, 2, \ldots, n$, and the integral of the residual over each subdomain is then required to be zero:

$$\int_{V_i} R\,dV_i = 0, \quad i = 1, 2, \ldots\ldots n$$

- This yields n simultaneous algebraic equations for the n unknowns Ci, $i = 1, 2, \ldots, n$. It can be seen that the method is equivalent to choosing

$$f(R) = R \text{ and } w = \begin{cases} 1 & \text{If } x \text{ is in } V_i \\ 0 & \text{If } x \text{ is not in } V_i \end{cases}$$

Problems in Sub Domain Collocation Method

- Find the solution of the differential equation $\frac{d^2\phi}{dx^2}+\phi+x=0; \quad 0\le x\le 1$. Subjected to the boundary conditions $\phi(0)=\phi(1)$ using sub domain collocation method, find the solution of the differential equation.

Approach: Assume an approximate solution satisfying the boundary conditions with two unknown constants. Set the integration of residue equal to zero at the collocation points to evaluate the constants

Solution:

The approximate solution satisfying the boundary conditions is taken as,

$$\bar{\phi}(x)=c_1x(1-x)+c_2x^2(1-x)$$

Problems in Sub Domain Collocation Method

where C1 and C2 are unknown constants. Using this solution, the residue can be expressed as,

$$R=\frac{d^2\bar{\phi}}{dx^2}+\bar{\phi}+x=c_1(-2+x-x^2)+c_2(2-6x+x^2-x^3)+x$$

$$\int_{x=0}^{\frac{1}{4}} R(x)dx=\left|c_1(-2x+\frac{x^2}{2}-\frac{x^3}{3})+c_2(2x-3x^2+\frac{x^3}{3}-\frac{x^4}{4})+\frac{x^2}{2}\right|_0^{\frac{1}{4}}$$

$$=c_1\left(-\frac{1}{2}+\frac{1}{32}-\frac{1}{192}\right)+c_2\left(\frac{1}{2}-\frac{3}{16}+\frac{1}{192}-\frac{1}{1024}\right)+\frac{1}{32}$$

$$=-\frac{91}{192}c_1+\frac{973}{3072}c_2+\frac{1}{32}=0$$

Problems in Sub Domain Collocation Method

$$\int_{x=\frac{1}{4}}^{\frac{1}{2}} R(x)dx = \left| c_1(-2x+\frac{x^2}{2}-\frac{x^3}{3})+c_2(2x-3x^2+\frac{x^3}{3}-\frac{x^4}{4})+\frac{x^2}{2}\right|_{\frac{1}{4}}^{\frac{1}{2}}$$

$$=c_1\left(-1+\frac{1}{8}-\frac{1}{24}\right)+c_2\left(1-\frac{3}{4}+\frac{1}{24}-\frac{1}{64}\right)+\frac{1}{8}$$

$$-\left\{c_1\left(-\frac{1}{2}+\frac{1}{32}-\frac{1}{192}\right)+c_2\left(\frac{1}{2}-\frac{1}{16}+\frac{1}{192}-\frac{1}{1024}\right)+\frac{1}{32}\right\}$$

$$=-\frac{85}{192}c_1+\frac{1661}{3072}c_2+\frac{3}{32}=0$$

The above equation can be written as ,can be rewritten as

$$-1456c_1+973c_2=-96$$

$$-1360c_1-1661c_2=-288$$

The final solution of the given problem

$$\bar{\phi}(x)=0.1175x(1-x)+0.0772x^2(1-x)$$

Galerkin Approach

- Here, the weights W_i are chosen to be the known functions $f(x)$ of the trial solution and the following n integrals of the weighted residual are set equal to zero:

$$\int_{V_i} f_i R dV = 0, \quad i = 1, 2, \ldots\ldots n$$

Problems Based on Galerkin Approach

- Find the solution of the differential equation $\frac{d^2\phi}{dx^2} + \phi + x = 0; \quad 0 \le x \le 1$

Subjected to the boundary conditions $\phi(0) = \phi(1)$ using a Galerkin method, find the solution of the differential equation.

Approach: Assume an approximate solution satisfying the boundary conditions with two unknown constants. Set integration of the weighted residue equal to zero at the collocation points to evaluate the constants

Solution:

The approximate solution satisfying the boundary conditions is taken as,

$$\bar{\phi}(x) = c_1 x(1-x) + c_2 x^2 (1-x)$$

Problems Based on Galerkin Approach

The integration of the elements has been done using Galerkin approach

$$\int_0^1 R(x)dx = \frac{1}{2} - \frac{11}{6}c_1 - \frac{11}{12}c_2$$

$$\int_0^1 xR(x)dx = \frac{1}{3} - \frac{11}{12}c_1 - \frac{19}{20}c_2$$

$$\int_0^1 x^2R(x)dx = \frac{1}{4} - \frac{37}{60}c_1 - \frac{4}{5}c_2$$

$$\int_0^1 x^3R(x)dx = \frac{1}{5} - \frac{7}{15}c_1 - \frac{17}{105}c_2$$

After solving those equations we will get,

$$-0.3c_1 - 0.15c_2 = -0.0833$$

$$-0.15c_1 - 0.1238c_2 = -0.05$$

The final result will be,

$$\bar{\phi}(x) = 0.1924x(1-x) + 0.1780x^2(1-x)$$

Problems on Least Squares Method

❖ Find the solution of the differential equation $\frac{d^2\phi}{dx^2}+\phi+x=0; \quad 0\le x\le 1$

Subjected to the boundary conditions $\phi(0)=\phi(1)$ using a Least square method,

Find the solution of the differential equation.

Approach: Assume an approximate solution satisfying the boundary conditions with two unknown constants. Set integration of the residue square equal to zero at the collocation points to evaluate the constants

Solution:

The approximate solution satisfying the boundary conditions is taken as,

$$\bar{\phi}(x)=c_1x(1-x)+c_2x^2(1-x)$$

Problems on Least Squares Method

The residue can be expressed as,

$$R = \frac{d^2\bar{\phi}}{dx^2} + \bar{\phi} + x = c_1(-2 + x - x^2) + c_2(2 - 6x + x^2 - x^3) + x$$

The integration of the square of the residue is,

$$I = \int_{x=0}^{1} w(x) R^2(x)\, dx$$

By assuming weighting function is unity,

$$I = \int_{x=0}^{1} R^2(x)\, dx$$

Problems on Least Squares Method

For the minimum value of the integral,

$$\frac{\partial I}{\partial c_1} = \int_0^1 2R \frac{\partial R}{\partial c_1} dx = 0$$

$$\frac{\partial I}{\partial c_2} = \int_0^1 2R \frac{\partial R}{\partial c_2} dx = 0$$

$$\int_0^1 R \frac{\partial R}{\partial c_1} dx = \int_0^1 \left[c_1 \left(-2 + x - x^2 \right) + c_2 \left(2 - 6x + x^2 - x^3 \right) + x \right] \left(-2 + x - x^2 \right) dx$$

$$= \frac{101}{30} c_1 + \frac{101}{60} c_2 - \frac{11}{12} = 0$$

$$\int_0^1 R \frac{\partial R}{\partial c_2} dx = \int_0^1 \left[c_1 \left(-2 + x - x^2 \right) + c_2 \left(2 - 6x + x^2 - x^3 \right) + x \right] \left(-2 - 6x + x^2 - x^3 \right) dx$$

$$= \frac{707}{420} c_1 + \frac{1572}{420} c_2 - \frac{399}{420} = 0$$

The final result will be,

$$\bar{\phi}(x) = 0.1875x(1 - x) + 0.1695(x)^2 (1 - x)$$

Discretization
of
Domain

Learning Objectives

At the end of this topic, you will be able to:

- Describe briefly the basic element shapes
- Explain in detail about discretization process
- Explain different node numbering schemes
- Describe the process of automatic mesh generation

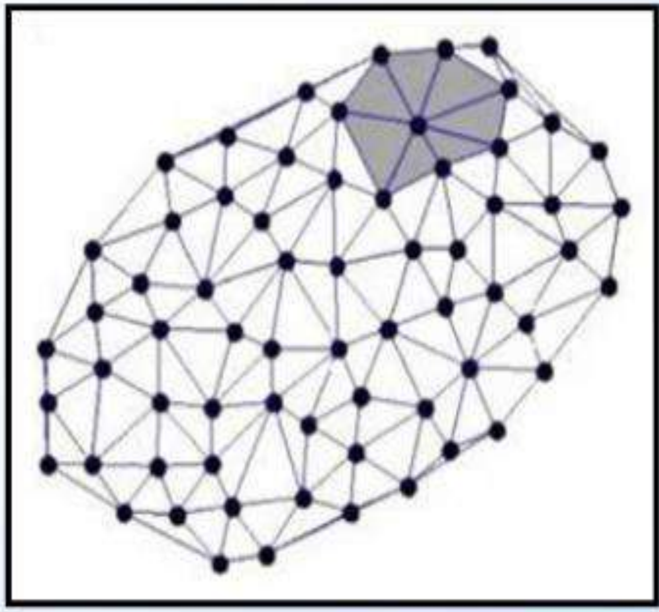
Outcomes

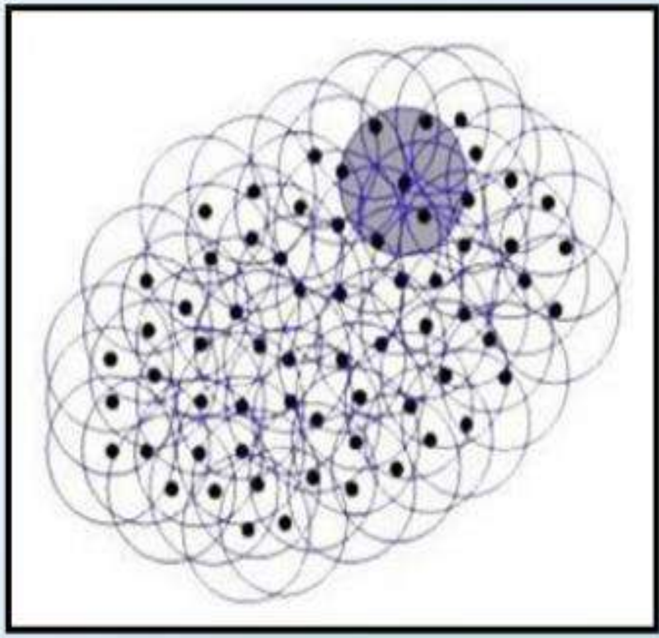
By the end of this topic, you will be able to:

- Understand the basic element shapes.
- Discuss about discretization process.
- Discuss about the different node numbering schemes.
- Understand the process of automatic mesh generation

Introduction

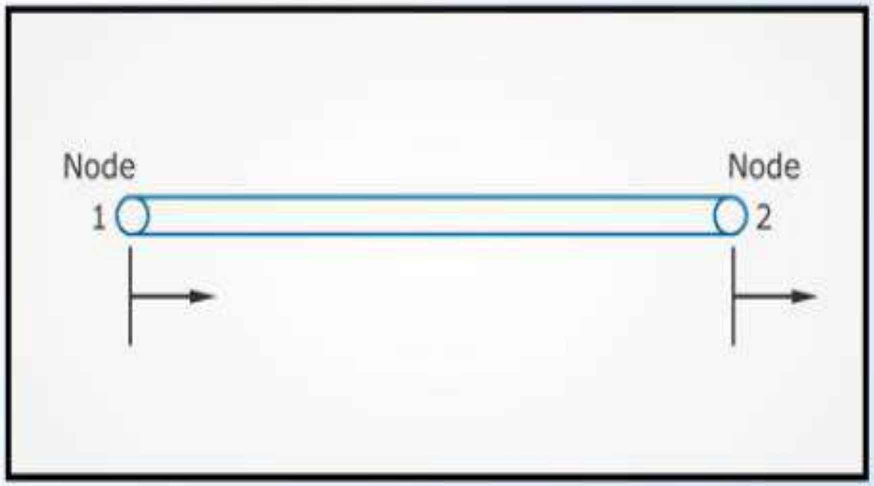
- Discretization of domain is the first step of finite element method.
- This involves discretization of irregular domains into smaller and regular sub-domains known as finite elements.
- This is equivalent to replacing the domain having an infinite number of degrees of freedom (dof) by a system having a finite number of dof.
- A variety of methods can be used to model a domain with finite elements.
- Different methods of dividing the domain into finite elements involve varying amounts of computational time and leads to different approximations to the solution of the physical problem.





Basic Element Shapes

- The shapes, sizes, number and configurations of the elements have to be chosen carefully such that the original body or domain is simulated as closely as possible without increasing the computational effort needed for the solution.
- The choice of the type of element is given by the geometry of the body and the number of independent coordinates necessary to describe the system
- If the geometry, material properties and the field variable of the problem is described in terms of a single spatial coordinate, we can use the one dimensional elements as shown.

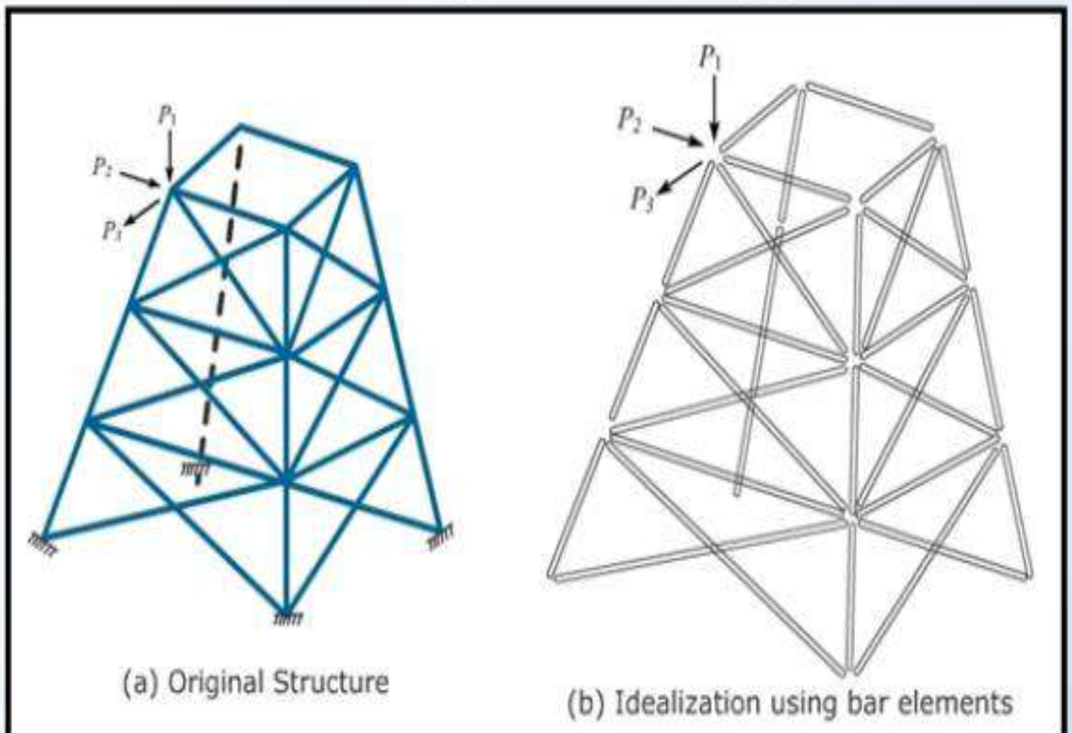


Discretization Process

Various considerations are to be taken into account in the discretization process.

Type of Elements:

- The type of elements to be used will be evident from the physical problem.
- If the problem involves the analysis of elements of truss structure under a given set of load conditions (as shown in (a)), the type of elements to be used for idealization is obviously the bar or line elements (as shown in (b)).

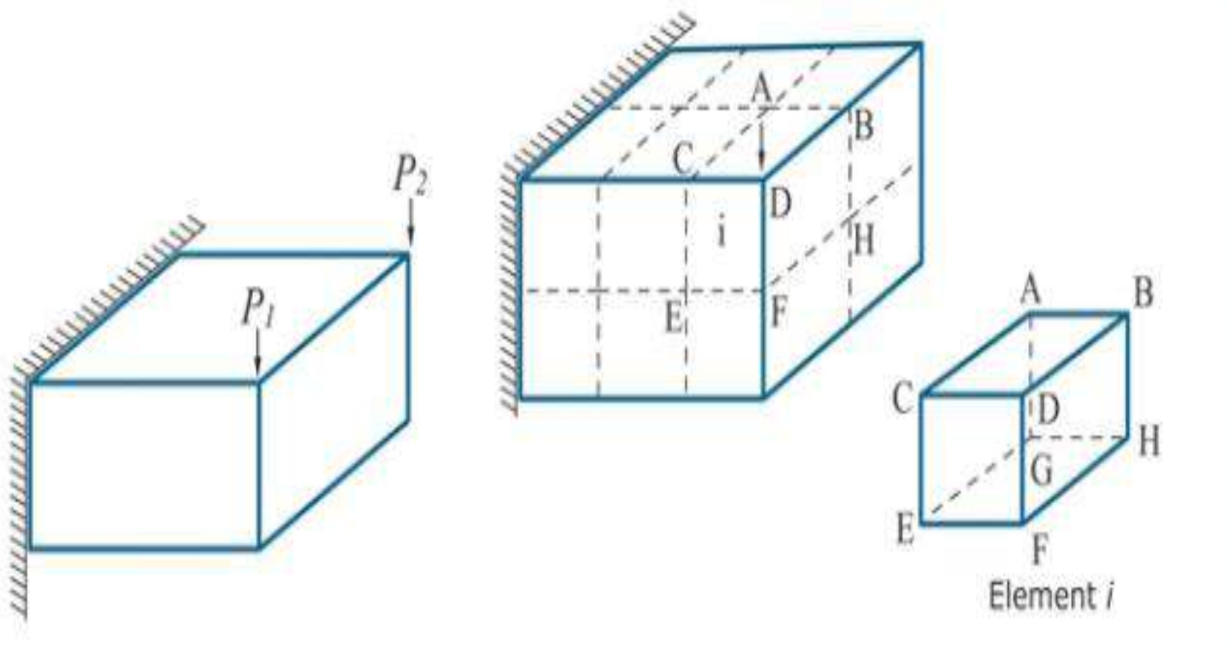


(a) Original Structure

(b) Idealization using bar elements

Discretization Process

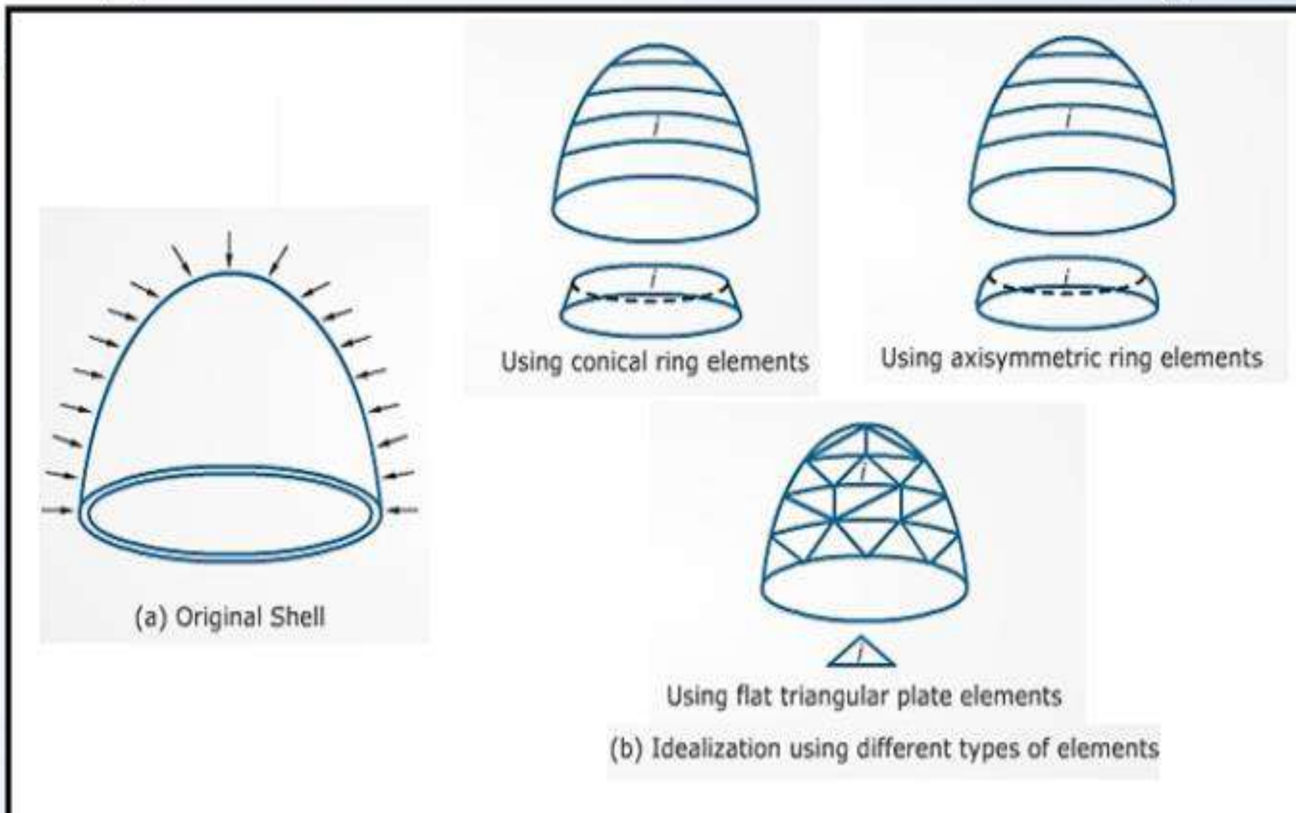
- In the case of stress analysis of the short beam (as shown in (a)) , the finite element idealization can be done using three-dimensional solid elements (as shown in (b)).



(a) Original Beam (b) Idealization using three-dimensional elements

Discretization Process

- Consider the problem of analysis of the thin walled shell (as shown in (a)).
- In this case, the shell can be idealized by several types of elements (as shown in (b)).
- Here, the number of dof needed, the expected accuracy, the ease with which the necessary equations can be derived, and the degree to which the physical structure can be modeled without approximation will dictate the choice of the element type to be used for idealization.

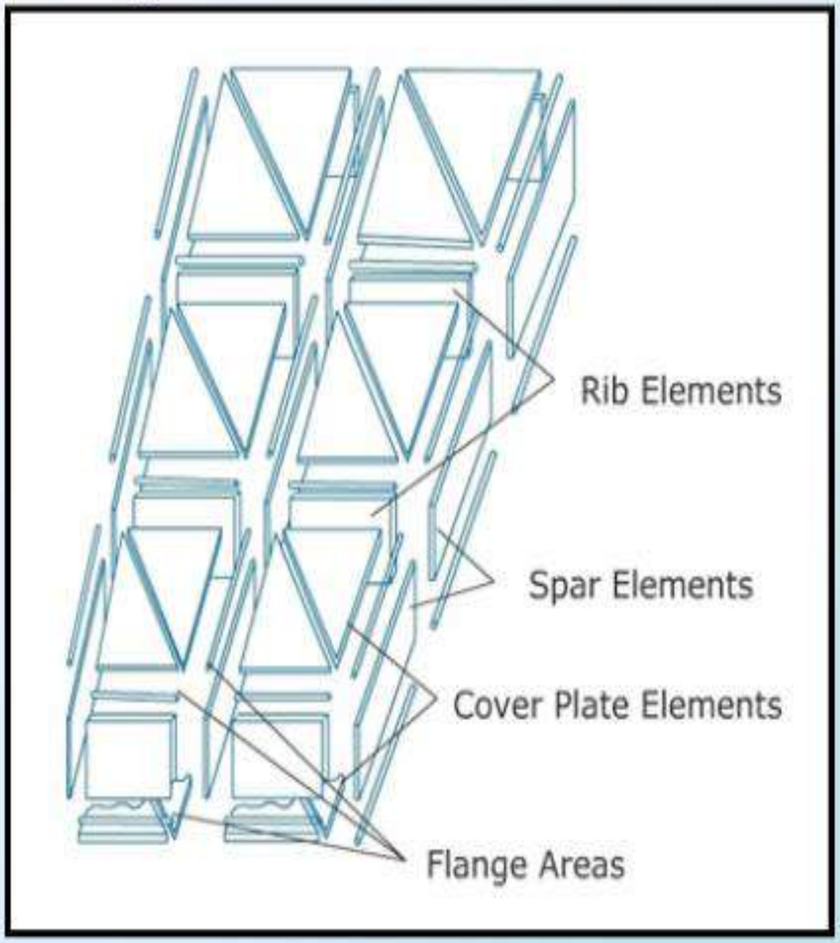


(a) Original Shell

(b) Idealization using different types of elements

Discretization Process

- In certain problems, the given body cannot be represented as an assemblage of only one type of elements.
- In such cases, we may have to use two or more types of elements for idealization. An example of this would be the analysis of an aircraft wing.
- Since the wing consists of top and bottom covers, stiffening webs, and flanges, three types of elements-namely, triangular plate elements (for covers), rectangular shear panels(for webs), and frame elements (for flanges) have been used in the idealization shown.



Example

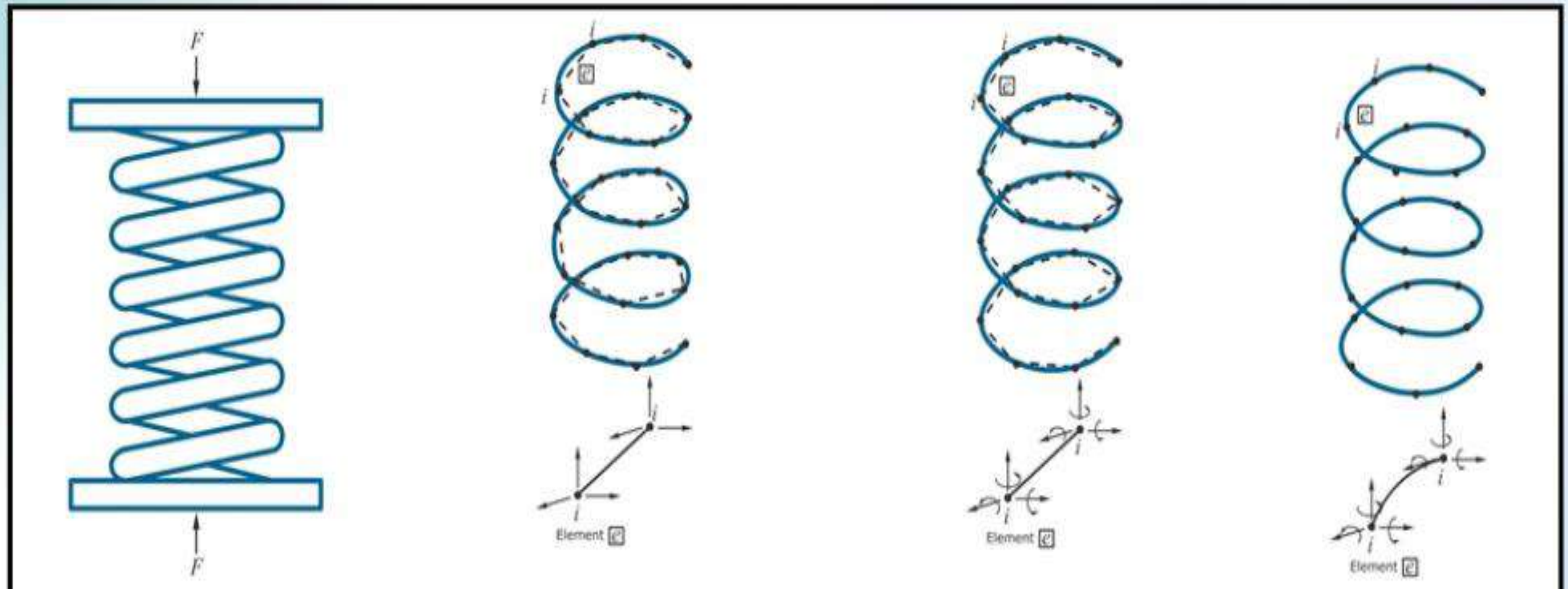
A helical spring is subjected to a compressive load as shown in Figure (a). Suggest different methods of modeling the spring using one-dimensional elements.

Solution:

Approach: Use various one-dimensional or line elements.

- The helical spring (in the form of curved wire) can be divided into several line or one-dimensional segments. These segments can be straight or curved.
- Each of the straight line segments (or elements) can be assumed to be a spatial truss element with each of its endpoints (or nodes) having three displacement dof (parallel to the x, y, and z axes) as shown in Figure (b).
- Since this element has only translational degrees of freedom (with no rotational degrees of freedom), it will not be able to carry any moment As such, the element may not be able to represent the behavior of the helical spring accurately.

Example



(a) Helical spring in compression (b) Spatial truss element (c) Spatial frame element (d) Curved frame element

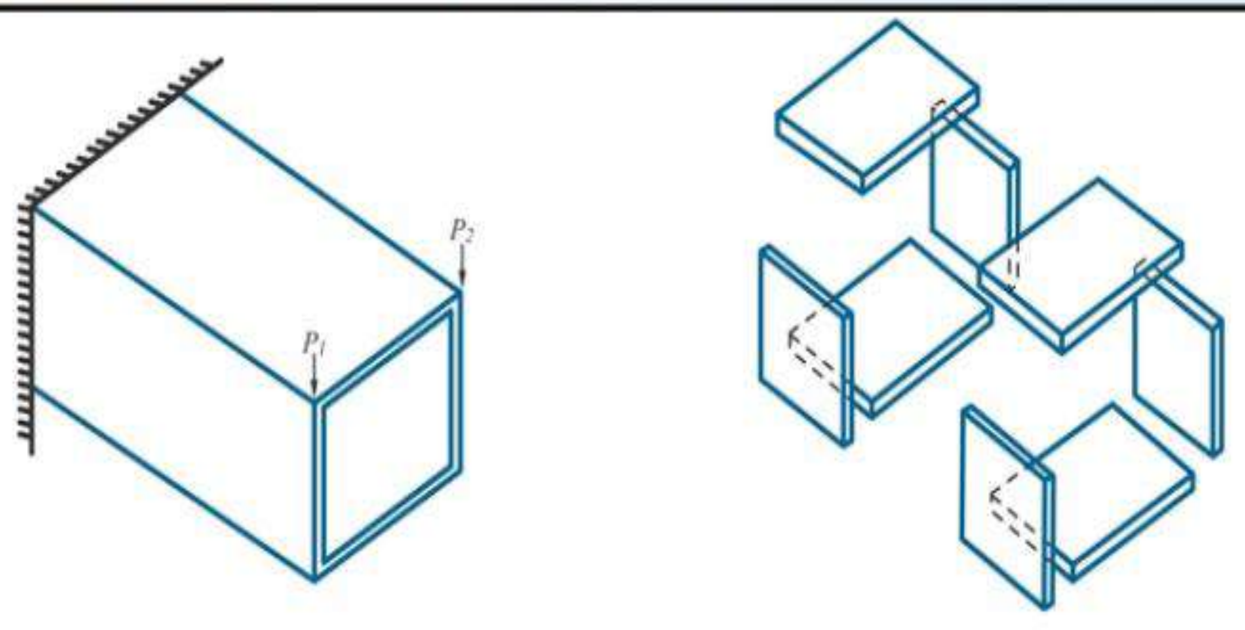
- Alternately, each of the straight line segments (or elements) can he assumed to be a spatial frame element with each of its endpoints (or nodes) having three displacement dof (parallel to the x, y, and z axes) and three rotational dof (about the x, y, and z axes) as shown in Figure (c).

Example

- In the case of the curved line segments (elements), each element can be treated as a curved frame element with three displacement dof (parallel to the x, y, and z axes) and three rotational dof (about the x, y, and z axes) at each end as shown in Figure (d).
- Because of the inclusion of rotational degrees of freedom, the models shown in Figures (c) and (d) will be able to simulate the behavior of the helical spring more accurately.

Size of Elements

- The size of elements influences the convergence of the solution directly, and hence it has to be chosen with care.
- If the size of the elements is small, the final solution is expected to be more accurate.
- The use of smaller-sized elements leads to more computation time. Sometimes, we have to use elements of different sizes in the same body.
- For example, in the case of stress analysis of the box beam shown in (a), the size of all the elements can be approximately the same, as shown in (b).

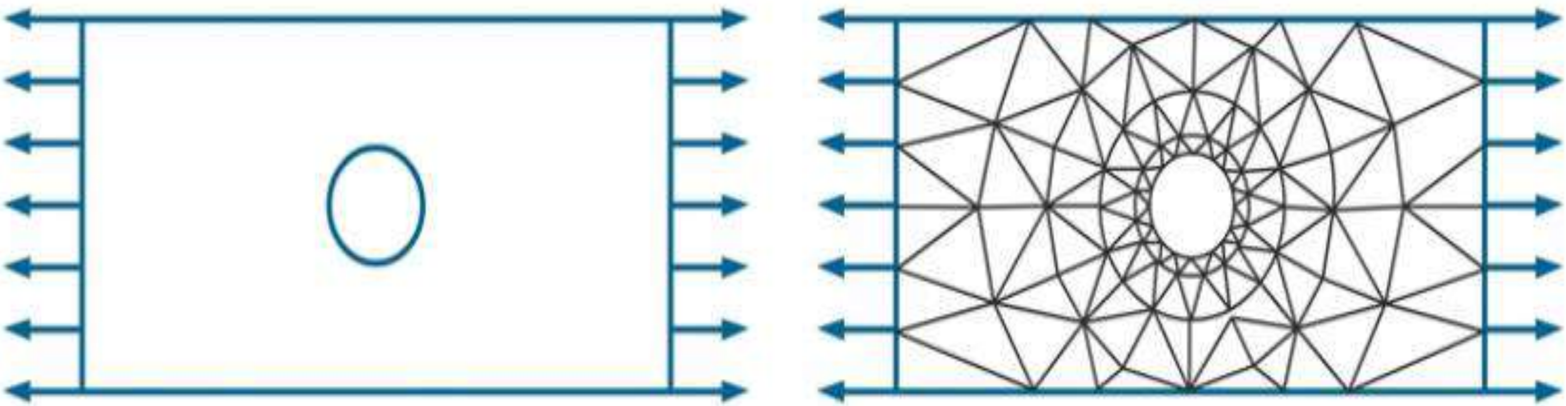


(a) Original structure

(b) Finite element idealization

Size of Elements

- In the case of stress analysis of a plate with a hole as shown in (a), the elements of different sizes have to be used, as shown in (b).
- The size of elements has to be very small near the hole compared to distant places.
- Another characteristic called aspect ratio, affects the finite element solution. It describes the shape of the element in the assemblage of elements.
- Elements with an aspect ratio of nearly unity generally yields best results.

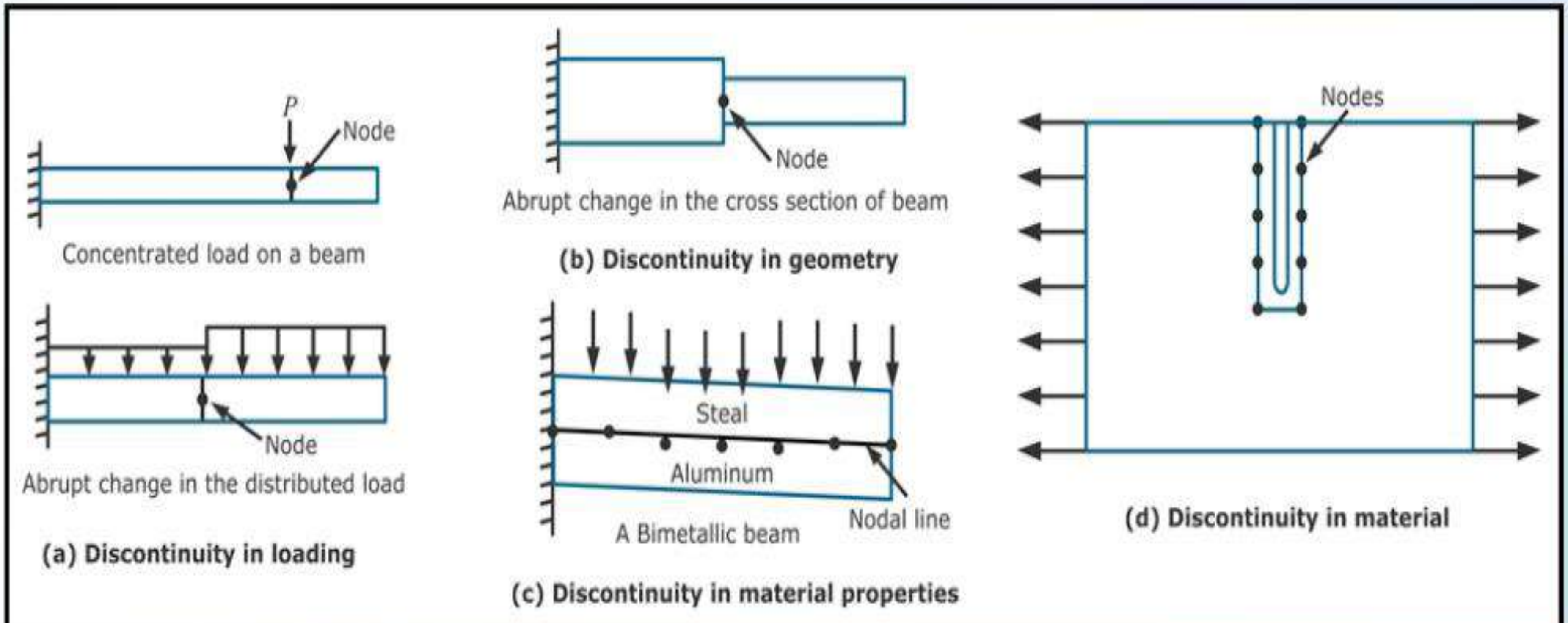


(a) Original structure

(b) Idealization using elements of different sizes

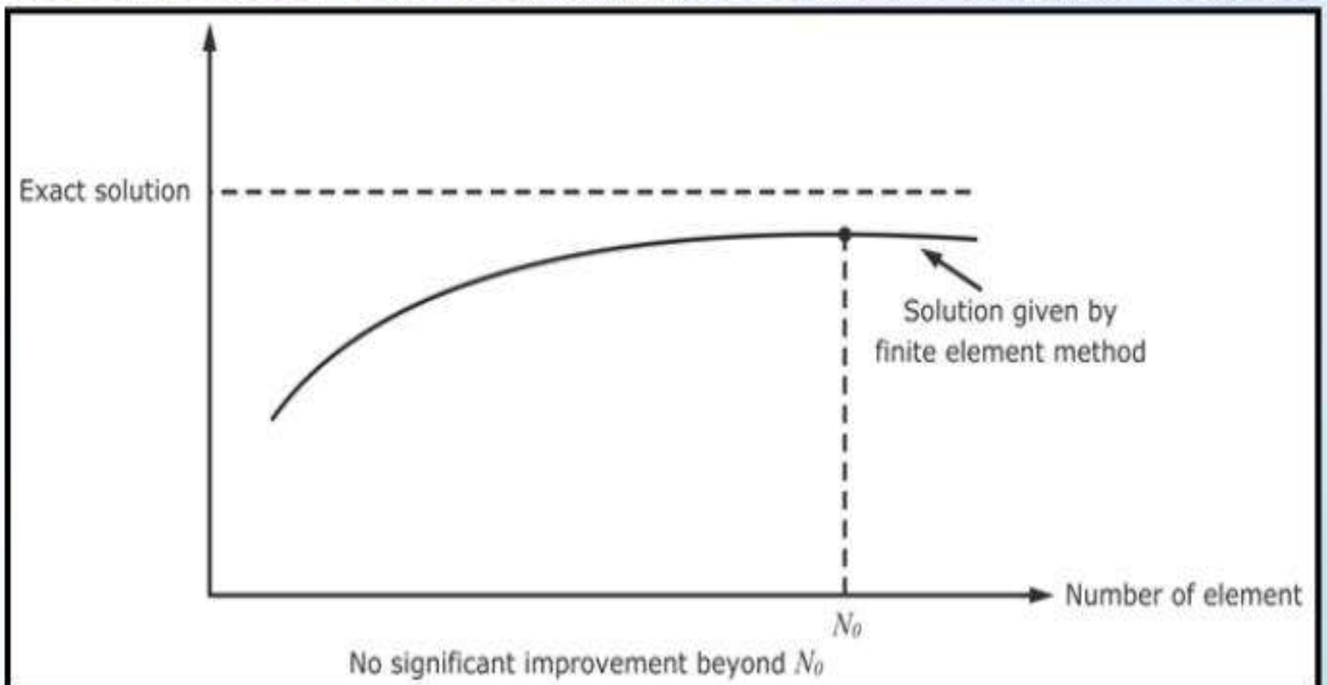
Location of Nodes

- If the body has no abrupt changes in geometry, material properties, and external conditions (e.g., load and temperature), the body can be divided into equal subdivisions and hence the spacing of the nodes can be uniform.
- On the other hand, if there are any discontinuities in the problem, nodes have to be introduced at these discontinuities, as shown.



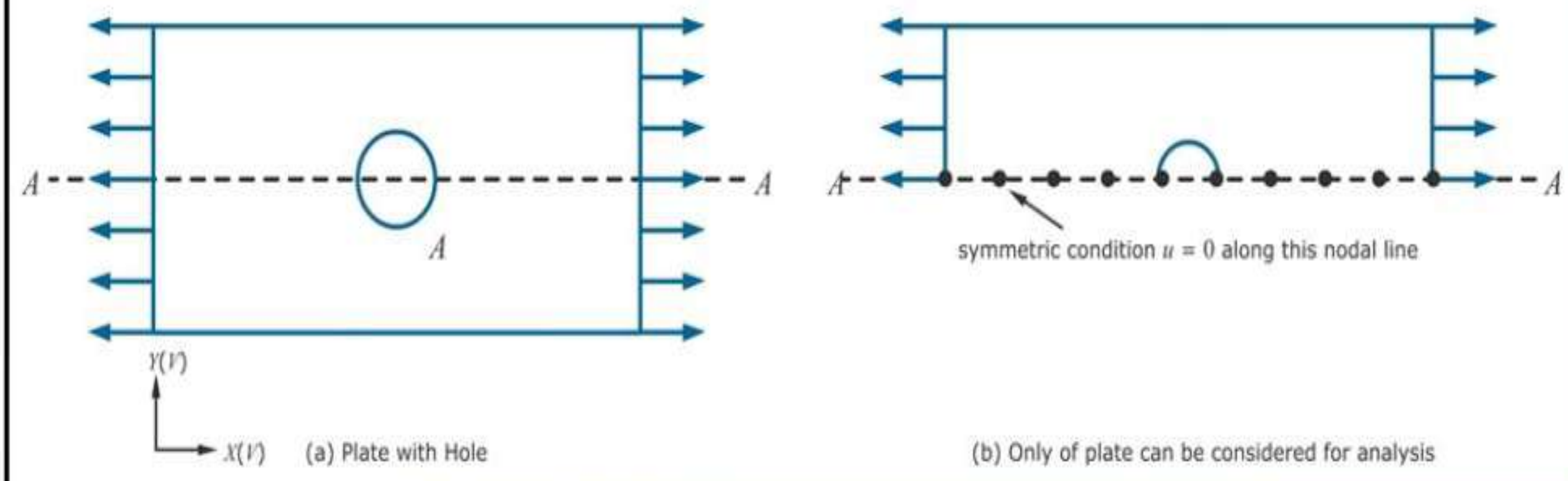
Number of Elements

- The number of elements to be chosen for idealization is related to the accuracy desired, size of elements, and the number of dof involved.
- Although an increase in the number of elements generally means more accurate results, for any given problem, there will be a certain number of elements beyond which the accuracy cannot be significantly improved. This behavior is shown graphically.
- Moreover, since the use of a large number of elements involves a large number of dof, we may not be able to store the resulting matrices in the available computer memory



Simplifications Afforded by the Physical Configuration of the Body

- If the configuration of the body as well as the external conditions are symmetric, we may consider only half of the body for finite element idealization.
- The symmetry conditions, however, have to be incorporated in the solution procedure. This is illustrated in figure, where only half of the plate with a hole, having symmetry in both geometry and loading, is considered for analysis.
- Since there cannot be a horizontal displacement along the line of symmetry AA, the condition that $u = 0$ has to be incorporated while finding the solution.

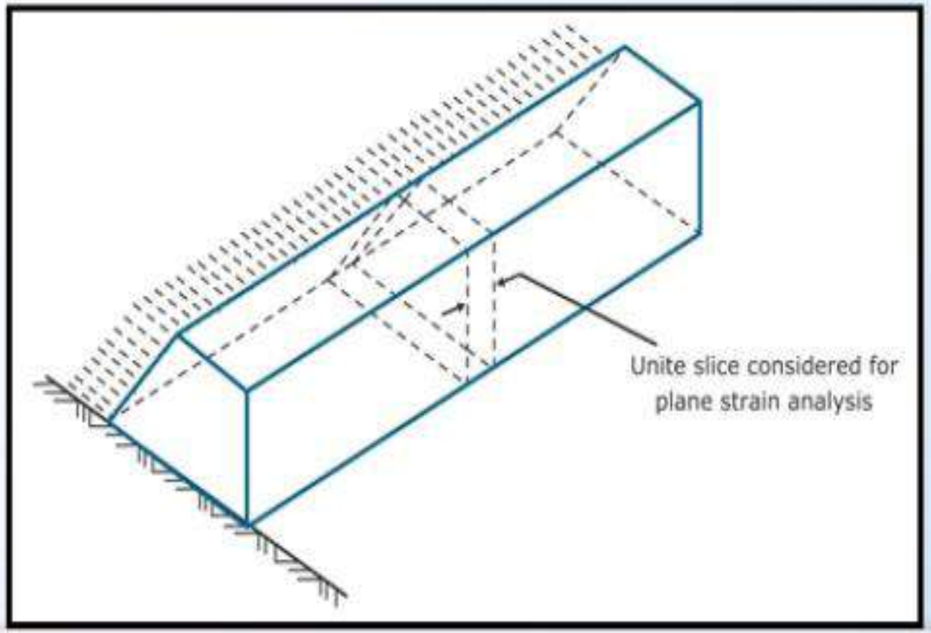


(a) Plate with Hole

(b) Only of plate can be considered for analysis

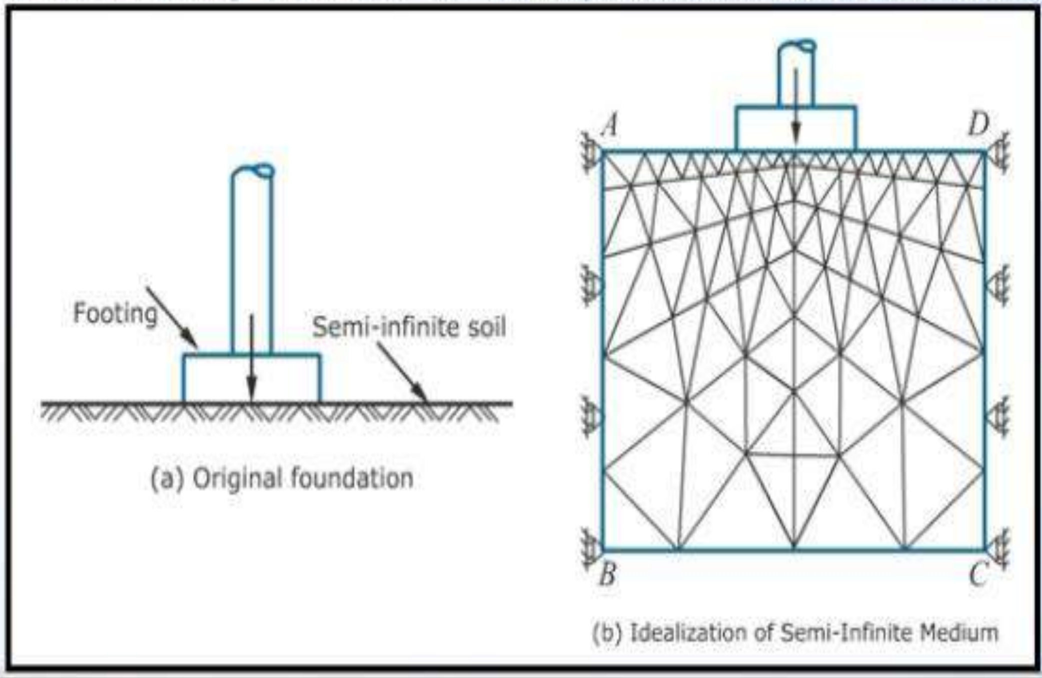
Finite Representation of Infinite Bodies

- In most of the problems, like in the analysis of beams, plates, and shells, the boundaries of the body or continuum are clearly defined. Hence, the entire body can be considered for element idealization.
- However, as in the analysis of dams, foundations, and semi-infinite bodies, the boundaries are not clearly defined.
- In the case of dams, since the geometry is uniform and the loading does not change in the length direction, a unit slice of the dam can be considered for idealization and analyzed as a plane strain problem.



Finite Representation of Infinite Bodies

- In the case of the foundation problem we cannot idealize the complete semi-infinite soil by finite elements.
- Since the effect of loading decreases gradually with increasing distance from the point of loading, we can consider only that much of the continuum in which the loading is expected to have a significant effect as shown in (b).
- Once the significant extent of the infinite body is identified as shown, the boundary conditions for this finite body have to be incorporated in the solution.

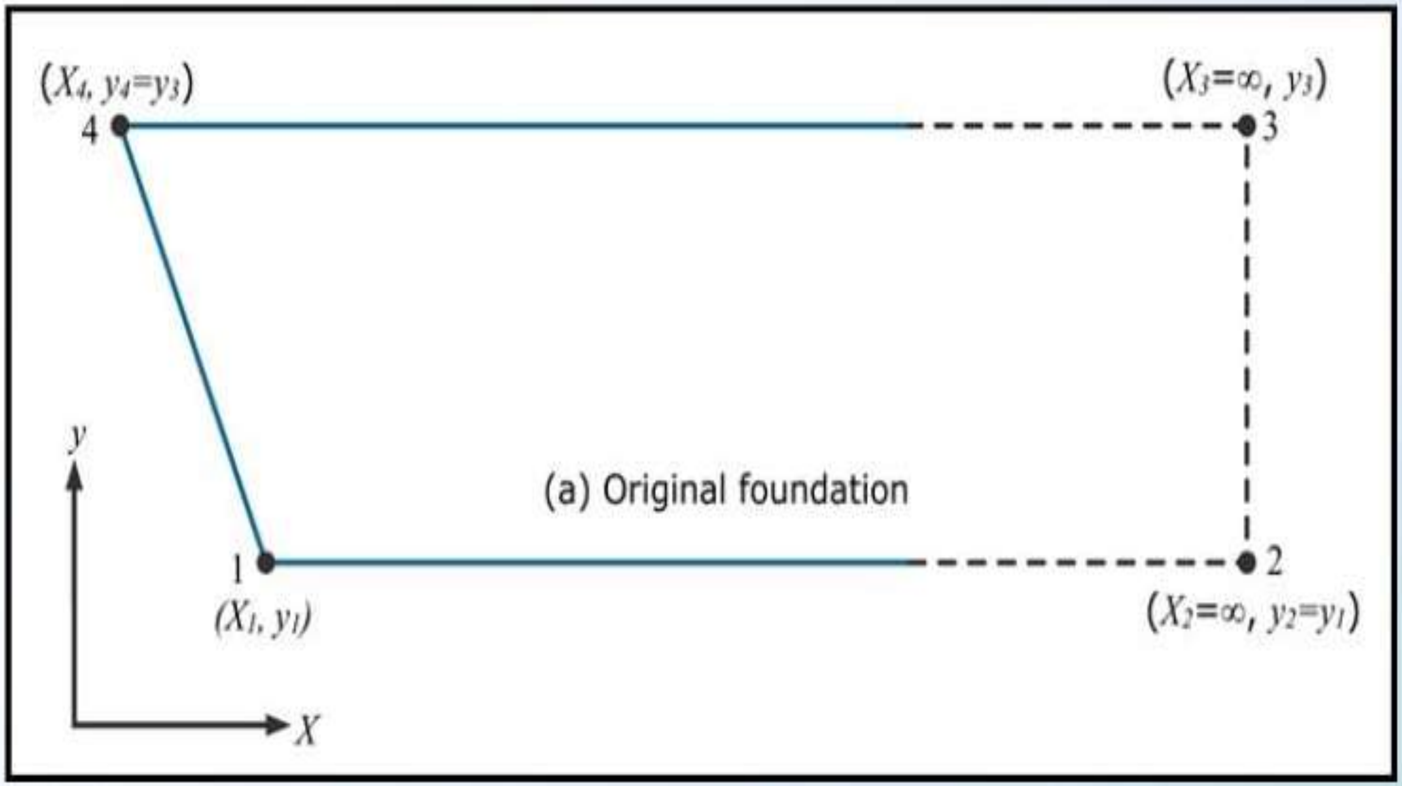


(a) Original foundation

(b) Idealization of Semi-Infinite Medium

Finite Representation of Infinite Bodies

- The semi-infinite soil has been simulated by considering only a finite portion of the soil.
- In some applications, the determination of the size of the finite domain may pose a problem.
- In such cases, one can use infinite elements for modeling.
- As an example, figure shows a four-node element that is infinitely long in the x direction.

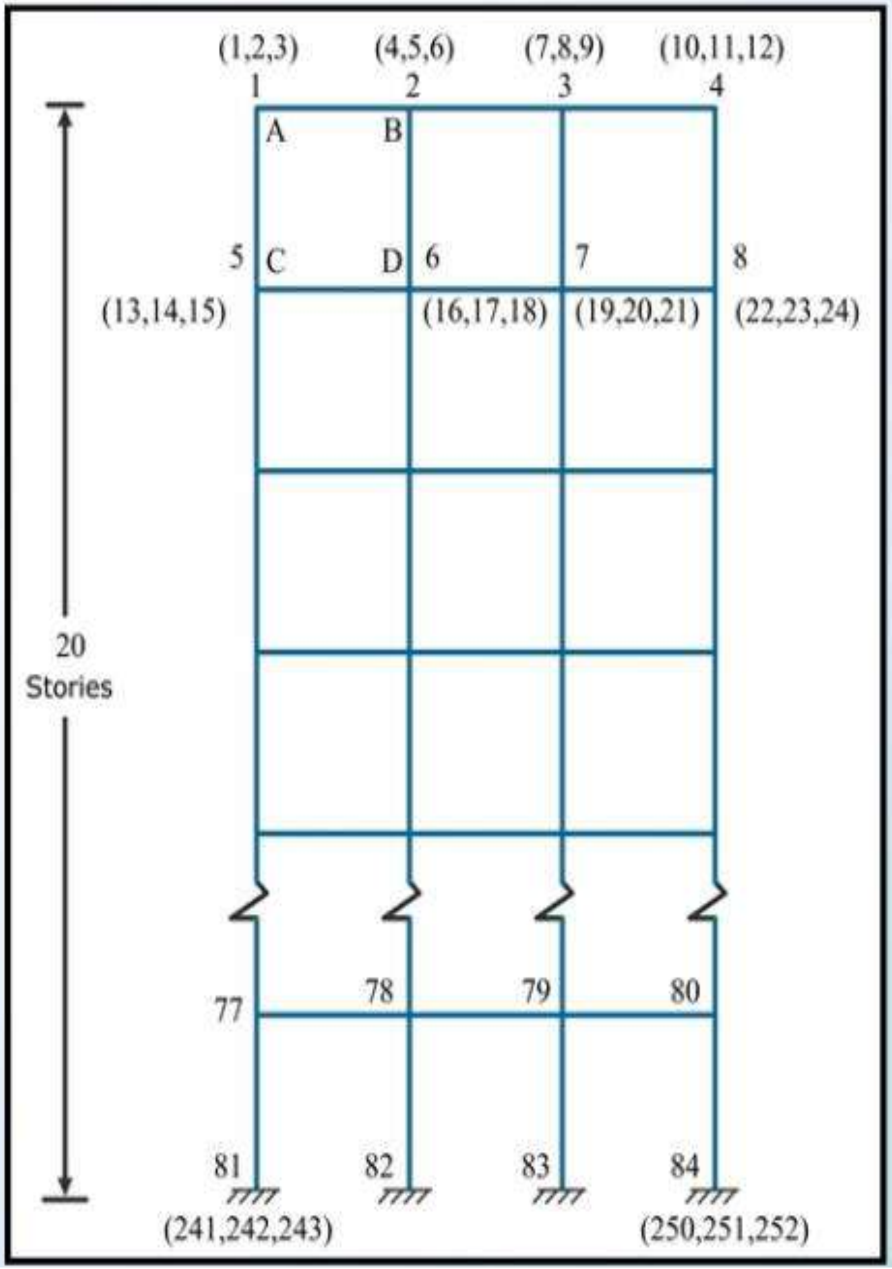


Node Numbering Scheme

- The finite element analysis of practical problems leads to matrix equations in which the matrices involved will be banded.
- Since most of the matrices involved are symmetric, the demands on the computer storage can be substantially reduced by storing only the elements involved in half bandwidth instead of storing the entire matrix.
- The bandwidth of the overall or global characteristic matrix depends on the node numbering scheme and the number of dof considered per node.
- If we can minimize the bandwidth, the storage requirements as well as solution time can also be minimized.

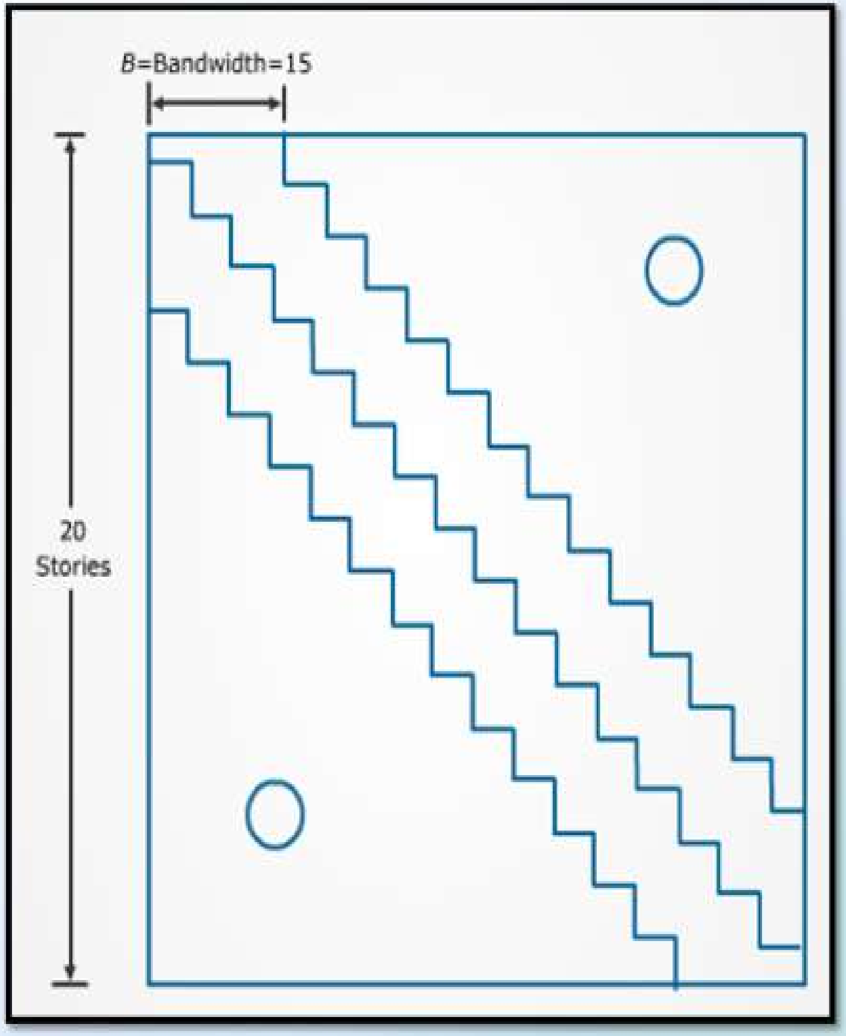
Node Numbering Scheme

- Consider a three-bay frame with rigid joints, 20 stories high, as shown.
- Assuming that there are 3 dof per node, there are 252 unknowns in the final equations (including the dof corresponding to the fixed nodes), and if the entire stiffness matrix is stored in the computer, it will require $252^2 = 63,504$ locations.
- The bandwidth (strictly speaking, half-bandwidth) of the overall stiffness matrix can be shown to be 15, and thus the storage required for the upper half-band is only 15 x 252 = 3780 locations.



Node Numbering Scheme

- By applying constraints to all the nodal dof except number 1 at node 1 (joint A), it is clear that an imposed unit displacement in the direction of 1 will require constraining forces at the nodes directly connected to node A-that is, B and C.
- These constraining forces are nothing but the cross-stiffness's appearing in the stiffness matrix, and these forces are confined to the nodes B and C.
- Thus, the nonzero terms in the first row of the global stiffness matrix will be confined to the first 15 positions.



Node Numbering Scheme

- The bandwidth is defined as

  B = (maximum difference between the numbered dof at the ends of any member+1)

- This definition can be generalized so as to be applicable for any type of finite element as

  Bandwidth $(B) = (D + 1).f$ - - - - - - -(1)

  Where, D is the maximum largest difference in the node numbers occurring for all elements of the assemblage and f is the number of dof at each node.

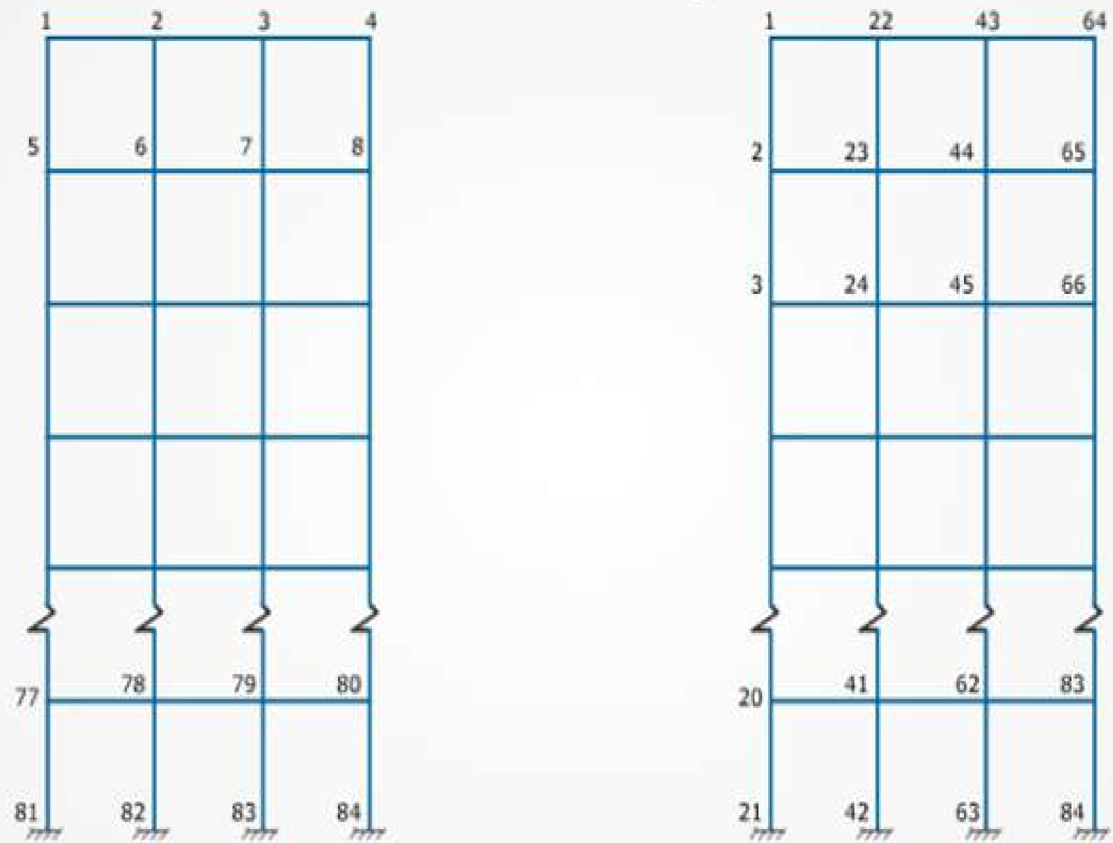
- The above equation indicates that D has to be minimized in order to minimize the bandwidth.

- Thus, a shorter bandwidth can be obtained simply by numbering the nodes across the shortest dimension of the body.

Node Numbering Scheme

❖ It is clear from the figure that the numbering of nodes along the shorter dimension produces a bandwidth of $B = 15$ $(D = 4)$, whereas the numbering along the longer dimension produces a bandwidth of $B = 66$ $(D = 21)$.

Different Node Numbering Schemes



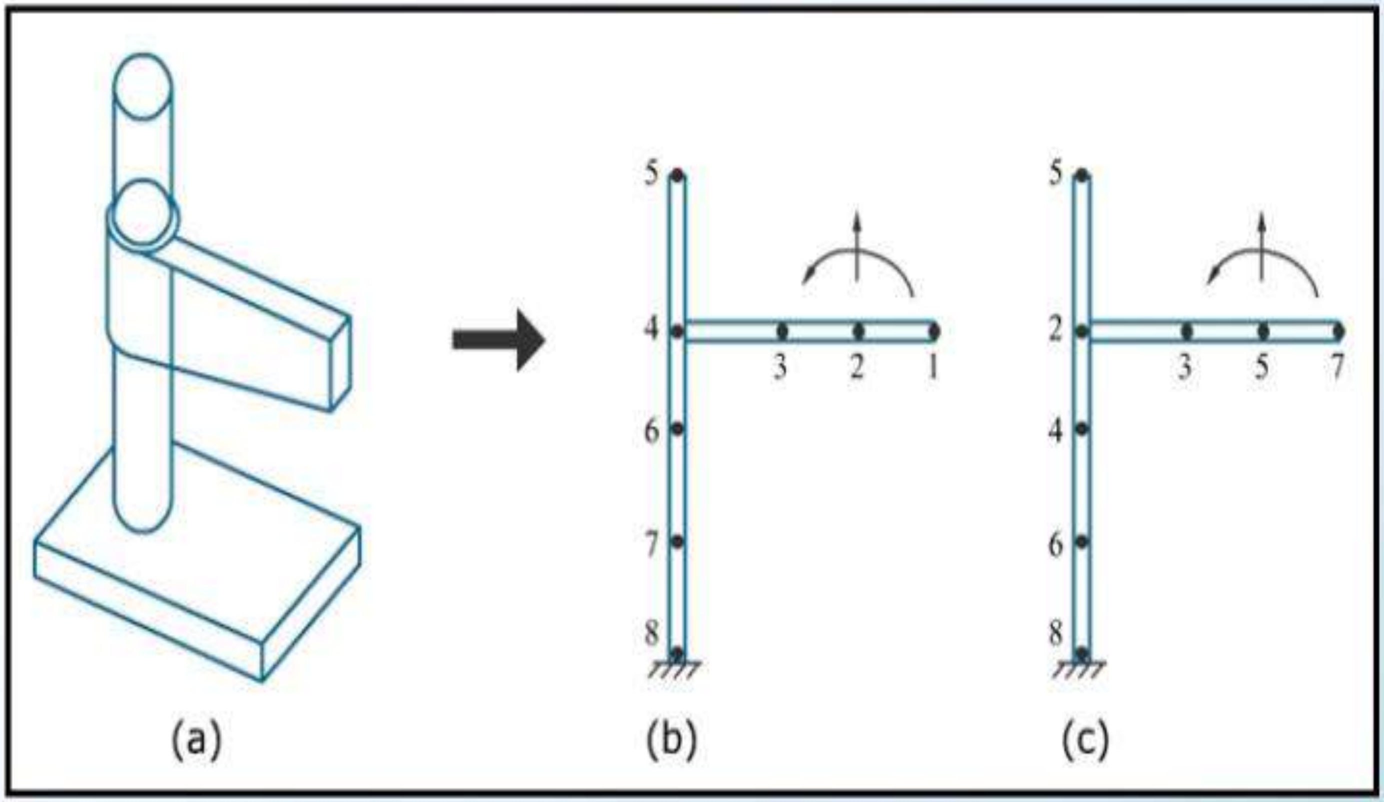
(a) Node Numbering along the Shorter Dimension (b) Node Numbering Scheme along the Longer Dimension

Example

A drilling machine is modeled using one-dimensional beam elements as shown in Figure(a). If two dof are associated with each node, label the node numbers for minimizing the bandwidth of the stiffness matrix of the system.

Solution

Approach: Number the nodes along the shorter side of the machine first.



(Contd.)

Example

- Because the column (vertical member) of the machine has 5 nodes and the arm (horizontal member) has only 4 nodes, we number the nodes along the shorter side as shown in Figure.
- Noting that the maximum difference between the numbers of the end nodes among all the elements is 2, the bandwidth of the resulting stiffness matrix of the system is given by

$$B = (D+1)f = (2+1)2 = 6$$

Note that the nodes can also be numbered as shown in Figure, which also yields the same bandwidth of $B = 6$.

Automatic Mesh Generation

- Mesh generation is the process of dividing a physical domain into smaller sub-domains (called elements) to facilitate an approximate solution of the governing ordinary or partial differential equation.
- For this, one-dimensional domains (straight or curved lines) are subdivided into smaller line segments, two-dimensional domains (planes or surfaces) are subdivided into triangle or quadrilateral shapes, and three-dimensional domains (volumes) are subdivided into tetrahedron and hexahedron shapes.
- If the physical domain is simple and the number of elements used is small, mesh generation can be done manually.
- The automatic mesh generation schemes are usually tied to solid modeling and computer aided design schemes.

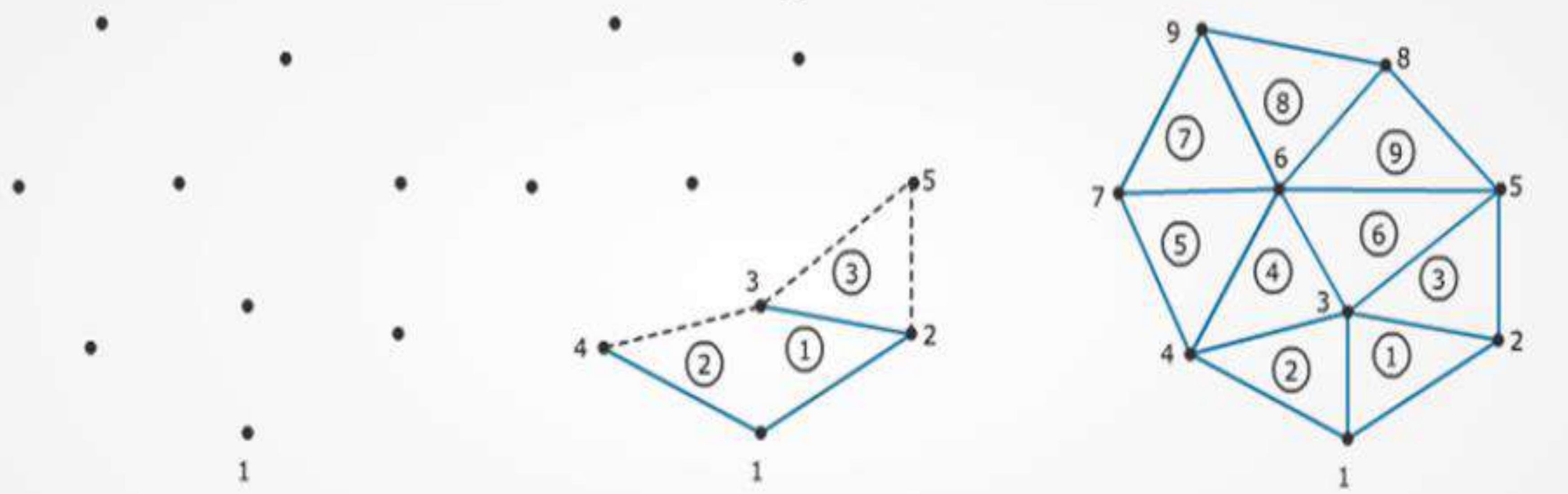
Automatic Mesh Generation

- When the user supplies information on the surfaces and volumes of the material domains that make up the object or system, an automatic mesh generator generates the nodes and elements in the object.
- The user can also specify minimum permissible element sizes for different regions of the object.
- Many mesh generation schemes first create all the nodes and then produce a mesh of triangles by connecting the nodes to form triangles (in a plane region).
- The most common methods used in the development of automatic mesh generators are the Tessellation and Octree methods.
- In the tesselation method, the user gives a collection of node points and also an arbitrary starting node.
- The method then creates the first simplex element using the neighboring nodes.

Automatic Mesh Generation

- Then a subsequent or neighboring element is generated by selecting the node point that gives the least distorted element shape.
- The procedure is continued until all the elements are generated.
- The step by - step procedure involved in this method is illustrated in figure for a two dimensional example.

Mesh Generation Using Tesselation Method



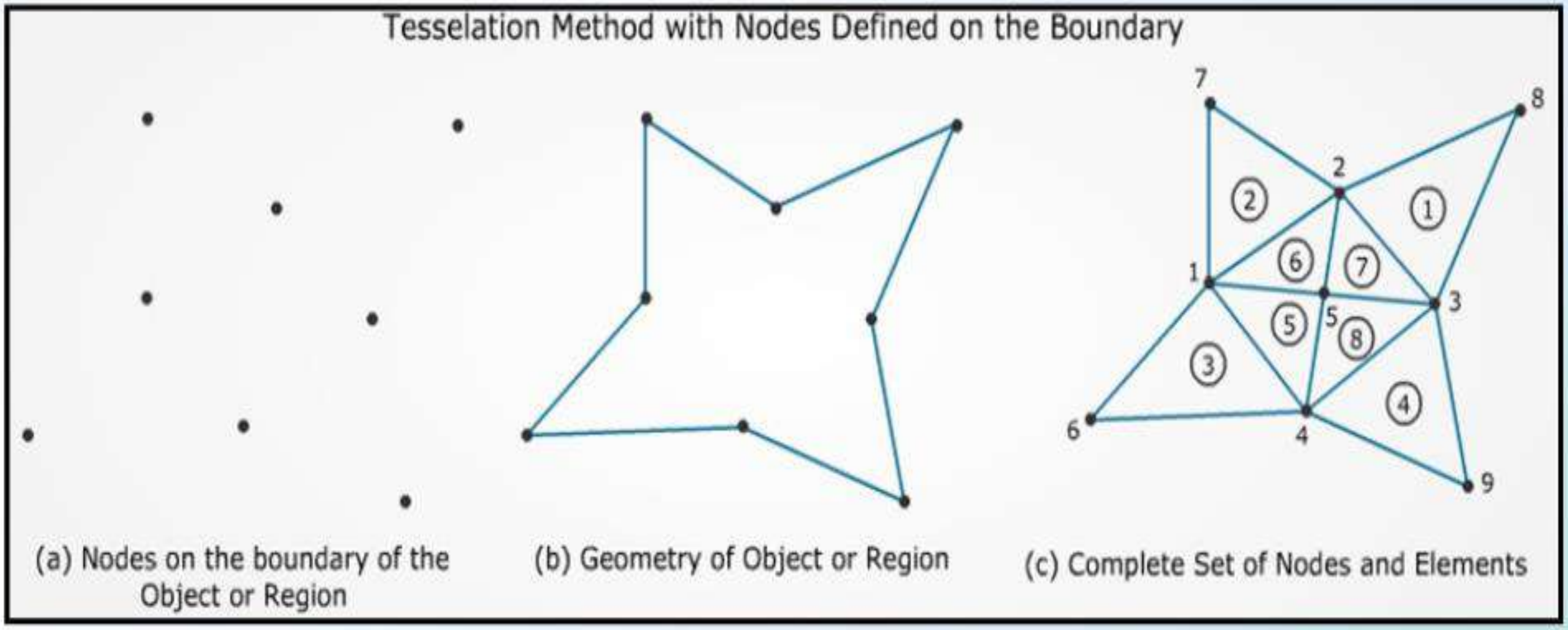
(a) Nodes in the Object or Region (b) Generation of Simplex Elements (c) Complete Set of Nodes and Elements

Automatic Mesh Generation

❖ The user can define the boundary of the object by a series of nodes.

❖ Then the tesselation method connects selected boundary nodes to generate simplex elements.

❖ The stepwise procedure used in this approach is shown.



Automatic Mesh Generation

- The octree methods belong to a class of mesh generation schemes known as tree structure methods, which are extensively used in solid modeling and computer graphics display methods.
- In the octree method, the object is first considered enclosed in a three dimensional cube.
- If the object does not completely (uniformly) cover the cube, the cube is subdivided into eight equal parts.
- In the two-dimensional analog of the octree method, known as the quad tree method, the object is first considered enclosed in a square region.
- If the object does not completely cover the square, the square is subdivided into four equal quadrants.
- If anyone of the resulting quadrants is full or empty, then it is not subdivided further.

Example

Generate the finite element mesh for the two-dimensional object (region) shown by the crossed lines in Figure (a) using the quadtree method.

Solution

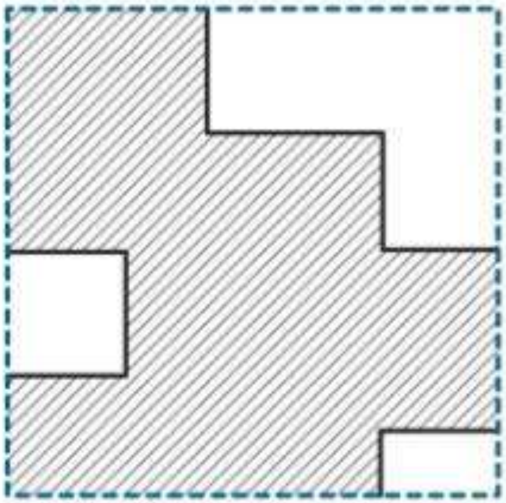
Approach: Use the quadtree method.

- First, the object is enclosed in a square region as shown by the dotted lines in Figure (a).
- Since the object does not occupy the complete square, the square is divided into four parts as shown in Figure (b).
- Since none of these parts are fully occupied by the object, each pan is subdivided into four parts as shown in Figure (c).

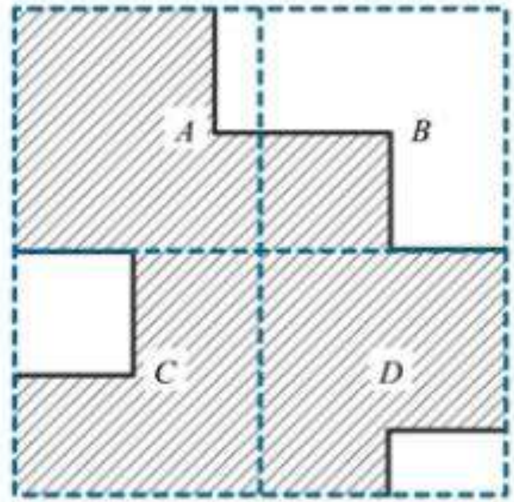
Example

- It can be seen that parts 1, 3, and 4 of A, part 3 of B, parts 2 to 4 of C, and parts 1 to 3 of D are completely occupied by the object, whereas pans 1, 2, and 4 of B and pan 1 of C are empty (not occupied by the object).
- In addition, part 2 of A and part 4 of D are partially occupied by the object; hence, they are further subdivided into four parts each as shown in Figure (d).
- It can be noted that parts a and y of pan 2 (of A) and parts a and b of part 4 (of D) are completely occupied while the remaining pans, namely b and s of pan 2 (of A) r and y of pan 4 (of D), are empty.
- Since all the parts at this stage are either completely occupied or completely empty, no further subdivision is necessary.
- The corresponding quadtree representation is shown in Figure (e). Note that the shape of the finite elements is assumed to be square in this example

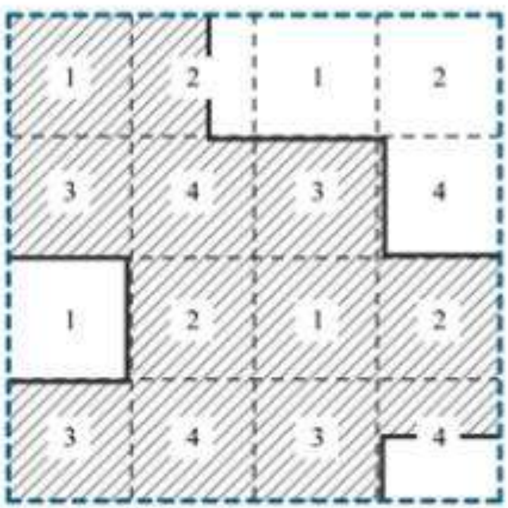
Example



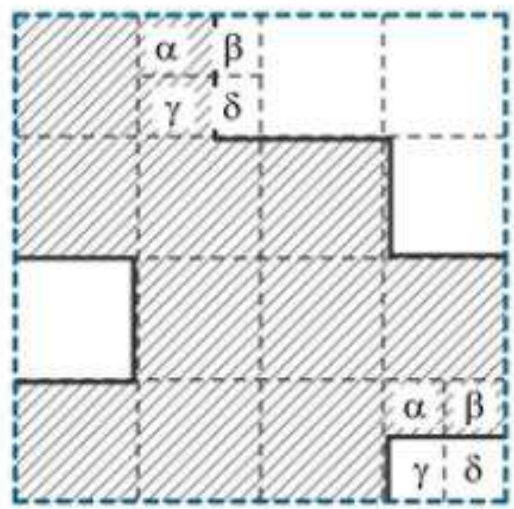
(a) Given Object Enclosed by a Square



(b) Division into four parts



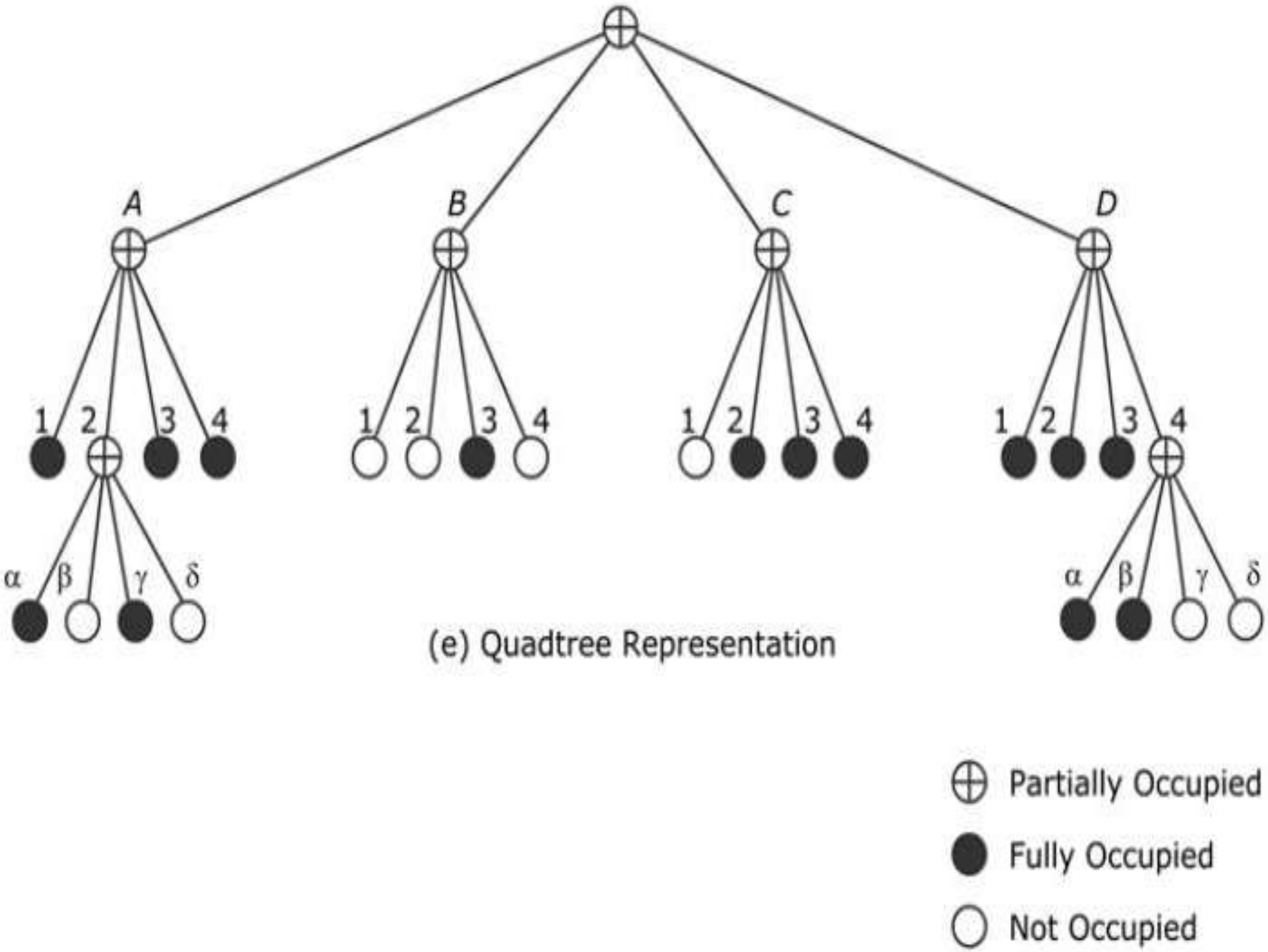
(c) Division of each partially occupied part into four parts



(d) Division of each partially occupied part into four parts

(Contd.)

Example



(e) Quadtree Representation

Interpolation Functions

Learning Objectives

At the end of this topic, you will be able to:

- Define interpolation function
- Reduce the polynomial equation to various cases of interest
- Classify finite elements based on geometry of the element and order of the polynomial
- Derive the linear interpolation polynomials for the basic one, two-and three-dimensional elements in terms of the global coordinates

Outcomes

By the end of this topic, you will be able to:

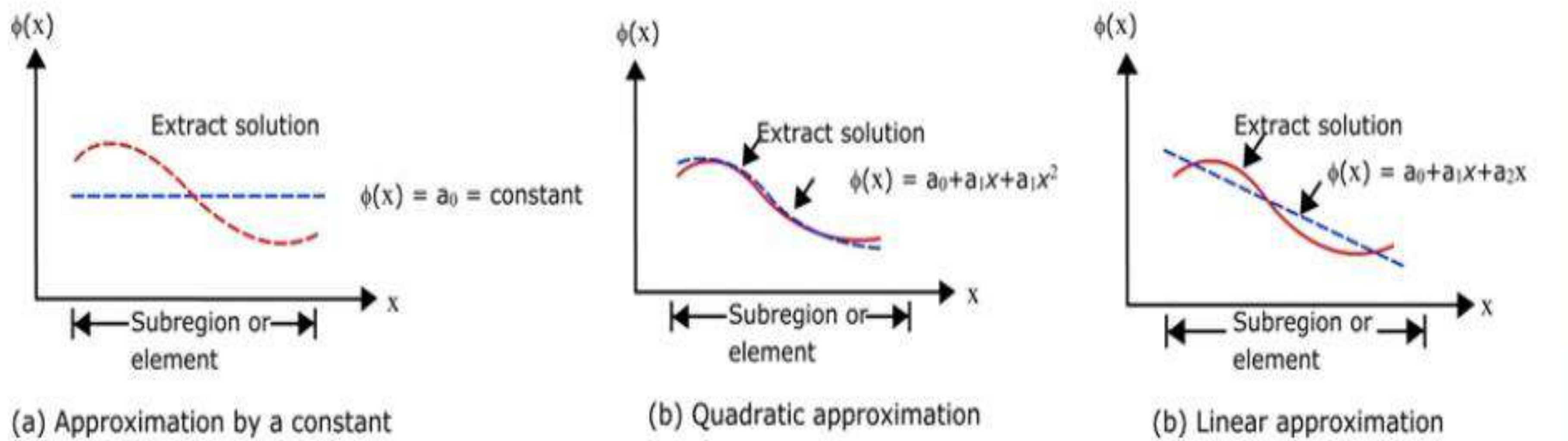
- Explain interpolation function
- Understand the polynomial equation to various cases of interest
- Understand the finite elements based on geometry of the element and order of the polynomial
- Discuss the linear interpolation polynomials for the basic one, two-and three-dimensional elements in terms of the global coordinates

Introduction

- The solution of a complicated problem in finite element method is obtained by dividing the region of interest into smaller regions i.e. finite elements and approximating the solution over each sub-region by a simple function.
- Thus, a necessary and important step is that of choosing a simple function for the solution in each element.
- The functions which are used to represent the behavior of the solution within an element are called interpolation functions or approximating functions or interpolation models.
- Polynomial type interpolation functions are most widely used because
  - It is easier to formulate and compute the finite element equations
  - To improve the accuracy of the results by increasing the order of the polynomial
- When the interpolation polynomial is of order one, the element is termed a linear element.

Introduction

- A linear element is called a simplex element if the number of nodes in the element is 2, 3, and 4 in one, two, and three dimensions, respectively.
- If the interpolation polynomial is of order two or more, the element is known as a higher order element.



Polynomial approximation in one dimension

Introduction

- In higher order elements, some secondary (mid-side and/or interior) nodes are introduced in addition to the primary (comer) nodes in order to match the number of nodal degrees of freedom with the number of constants (generalized coordinates) in the interpolation polynomial.
- If the order of the interpolation polynomial is fixed, the discretization of the region (or domain) can be improved by two methods.
- In the first method, known as the **r-method**, the locations of the nodes are altered without changing the total number of elements.
- In the second method, known as the **h-method**, the number of elements is increased.
- On the other hand, if improvement in accuracy is sought by increasing the order of the interpolation of polynomial the method is known as the **p-method**.

Polynomial Form of Interpolation Functions

- If a polynomial type of variation is assumed for the field variable $\varphi(x)$ in a one-dimensional element, $\varphi(x)$ can be expressed as

$$\phi(x) = \alpha_1 + \alpha_2 x + \alpha_3 x^2 + \ldots + \alpha_m x^n \quad \text{- - - - - (1)}$$

- Similarly in two and three dimensional finite elements the polynomial form of interpolation functions can be expressed as

$$\phi(x,y) = \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 x^2 + \alpha_5 y^2 + \alpha_6 xy + \ldots + \alpha_m y^n \quad \text{- - - - -(2)}$$

$$\phi(x,y,z) = \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 z + \alpha_5 x^2 + \alpha_6 y^2 + \alpha_7 z^2$$
$$+ \alpha_8 xy + \alpha_9 yz + \alpha_{10} zx + \ldots + \alpha_m z^n \quad \text{- - - - - (3)}$$

Where α_1 , α_2,..... α_μ are the coefficients of the polynomial, also known as generalized coordinates; n is the degree of the polynomial.

(contd.)

Polynomial Form of Interpolation Functions

- The number of polynomial coefficients m is given by

$$m = n + 1 \text{ for one-dimensional elements} \qquad \text{- - - - - - (4)}$$

$$m = \sum_{j=1}^{n+1} j \text{ for two-dimensional elements} \qquad \text{- - - - - - (5)}$$

$$m = \sum_{j=1}^{n+1} j(n+2-j) \text{ for three-dimensional elements} \qquad \text{- - - - - - (6)}$$

- In most practical applications, the order of the polynomial, n the interpolation functions is taken as one, two, or three.
- Thus, Equations (1) to (3) reduce to the following equations for various cases of practical interest.

Polynomial Form of Interpolation Functions

- For $n=1$ (linear model)

One-dimensional case: $\phi(x) = \alpha_1 + \alpha_2 x$ - - -(7)

Two-dimensional case: $\phi(x, y) = \alpha_1 + \alpha_2 x + \alpha_3 y$ - - -(8)

Three-dimensional case: $\phi(x, y, z) = \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 z$ - - -(9)

- For $n=2$ (quadratic model)

One-dimensional case: $\phi(x) = \alpha \ + \alpha_2 x + \alpha_3 x^2$ - - -(10)

Two-dimensional case: $\phi(x, y) = \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 x^2 + \alpha_5 y^2 + \alpha_6 xy$ - - - (11)

Three-dimensional case:
$$\phi(x, y, z) = \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 z + \alpha_5 x^2 + \alpha_6 y^2 + \alpha_7 z^2 + \alpha_8 xy + \alpha_9 yz + \alpha_{10} zx$$ - - -(12)

Polynomial Form of Interpolation Functions

- For $n=3$ (cubic model)

One-dimensional case:

$$\phi(x) = \alpha_1 + \alpha_2 x + \alpha_3 x^2 + \alpha_4 x^3 \quad \text{---(13)}$$

Two-dimensional case:

$$\phi(x,y) = \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 x^2 + \alpha_5 y^2 + \alpha_6 xy + \alpha_7 x^3 + \alpha_8 y^3 + \alpha_9 x^2 y + \alpha_{10} xy^2 \quad \text{---(14)}$$

Three-dimensional case:

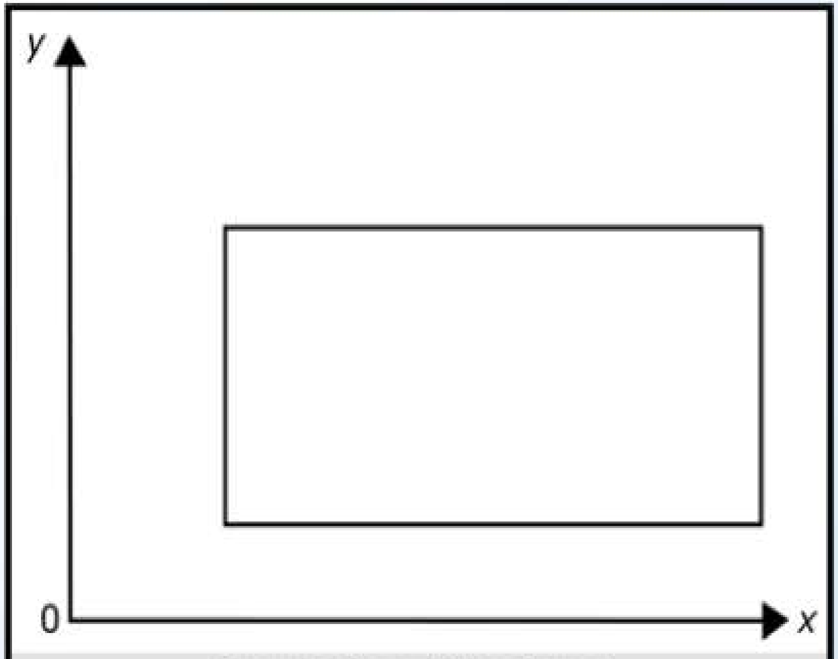
$$\begin{aligned}\phi(x,y,z) = {} & \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 z + \alpha_5 x^2 + \alpha_6 y^2 + \alpha_7 z^2 \\ & + \alpha_8 xy + \alpha_9 yz + \alpha_{10} zx + \alpha_{11} x^3 + \alpha_{12} y^3 + \alpha_{13} z^3 + \alpha_{14} x^2 y + \alpha_{15} x^2 z \quad \text{---(15)} \\ & + \alpha_{16} y^2 z + \alpha_{17} xy^2 + \alpha_{18} xz^2 + \alpha_{19} yz^2 + \alpha_{20} xyz\end{aligned}$$

Simplex, Complex and Multiplex Elements

- Based on the geometry of the element and the order of the polynomial used in the interpolation function, finite elements can be classified as simplex, complex and multiplex elements.
- The simplex elements are those for which the approximating polynomial consists of constant and linear terms.
- Thus, the polynomials given by equations 7 to 9 represent the simplex functions for one, two, and three-dimensional elements.
- For example, the simplex element in two dimensions is a triangle with three nodes (corners).
- The three polynomial coefficients α_1 , α_2, and α_3 of equation 8 can thus be expressed in terms of the nodal values of the field variable φ.
- The polynomials given by equations (10) to (15) denote complex functions.
- The complex elements may have the same shapes as the simplex elements but will have additional boundary nodes and, sometimes, internal nodes.

Simplex, Complex and Multiplex Elements

- For example, the interpolating polynomial for a two-dimensional complex element (including terms up to quadratic terms) is given by equation (11).
- Since this equation has six unknown coefficients α_i, the corresponding complex element must have six nodes.
- The multiplex elements are those whose boundaries are parallel to the coordinate axes to achieve interelement continuity, and whose approximating polynomials contain higher order terms.



Example of a multiplex element

Interpolation Polynomial in Terms of Nodal Degrees of Freedom

- The unknown solution or the field variable (e.g., displacement, pressure, or temperature) inside any finite element is assumed to be given by a simple function in terms of the nodal values of that element.
- The nodal values of the solution, also known as nodal degrees of freedom, are treated as unknowns in formulating the system or overall equations.
- The solution of the system equations gives the values of the unknown nodal degrees of freedom.
- Once the nodal degrees of freedom are known, the solution within any finite element (and hence within the complete body) will also be known to us.
- Thus, we need to express the approximating polynomial in terms of the nodal degrees of freedom of a typical finite element e.
- For this, let the finite element have M nodes.

Interpolation Polynomial in Terms of Nodal Degrees of Freedom

- We can evaluate the values of the field variable at the nodes by substituting the nodal coordinates into the polynomial equation given by equations (1) to (3). For example, equation (1) can be expressed as

$$\phi(x) = \vec{\eta}^{T} \vec{\alpha} \qquad ---(16)$$

where, $\vec{\eta}^{T} = \{1 \;\; x \;\; x^2 \ldots\ldots x^n\}$,

and $\vec{\alpha} = \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ . \\ . \\ . \\ \alpha_{n+1} \end{Bmatrix}$

Interpolation Polynomial in Terms of Nodal Degrees of Freedom

❖ The evaluation of equation (16) at the various nodes of element e gives

$$\begin{Bmatrix} \phi(at\ \text{node1}) \\ \phi(at\ \text{node1}). \\ . \\ . \\ . \\ \phi(at\ \text{node M}) \end{Bmatrix}^{(e)} = \overrightarrow{\Phi}^{(e)} = \begin{bmatrix} \vec{\eta}^{T}(at\ \text{node1}) \\ \vec{\eta}^{T}(at\ \text{node1}). \\ . \\ . \\ . \\ \vec{\eta}^{T}(at\ \text{node M}) \end{bmatrix} \vec{\alpha} \equiv \left[\underline{\eta}\right]\vec{\alpha} \qquad \text{---(17)}$$

❖ By inverting equation (17) , we get

-

$$\vec{\alpha} = \left[\underline{\eta}\right]^{-1} \overrightarrow{\Phi}^{(e)} \qquad \text{--- (18)}$$

Interpolation Polynomial in Terms of Nodal Degrees of Freedom

- Substituting equation(18) into equations (1) to (3) gives

$$\phi = \vec{\eta}^T \vec{\alpha} = \vec{\eta}^T \left[\underline{\eta}\right]^{-1} \overrightarrow{\Phi}^{(e)} = [N] \overrightarrow{\Phi}^{(e)} \qquad \text{--- (19)}$$

where

$$[N] = \vec{\eta}^T \left[\underline{\eta}\right]^{-1} \qquad \text{---(20)}$$

- Equation (19) expresses the interpolating polynomial inside any finite element in terms of the nodal unknowns of that element $\vec{\phi}^{(e)}$

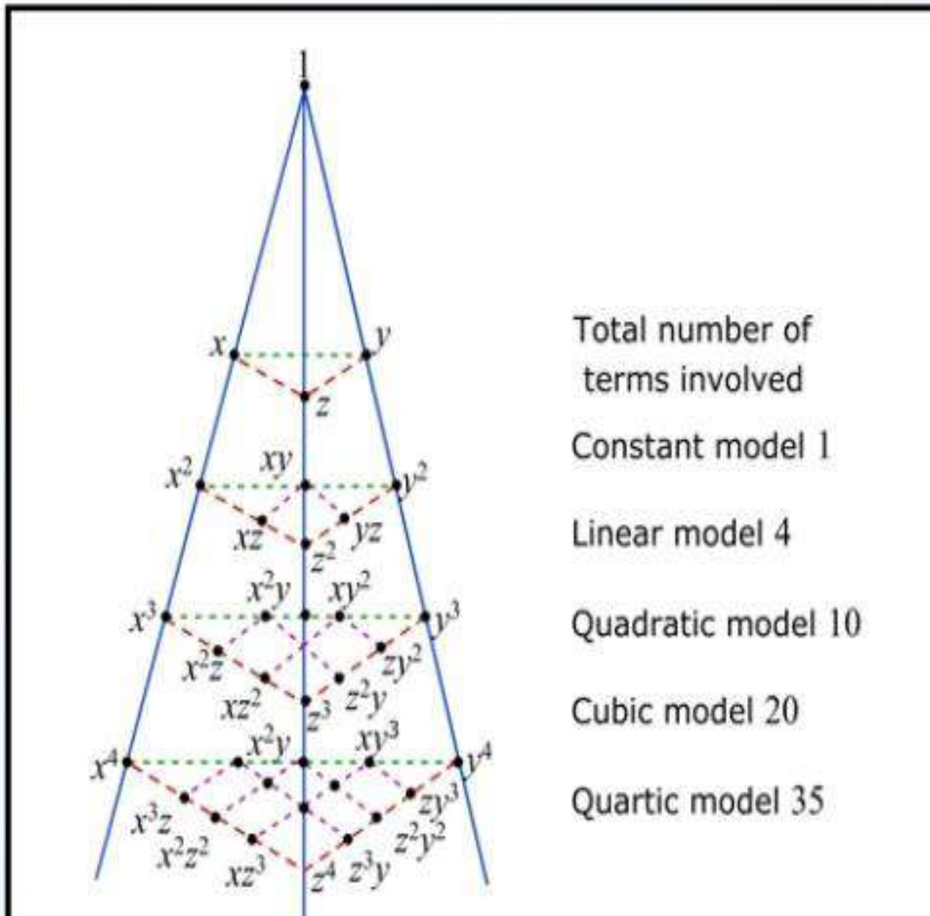
Selection of the Order of the Interpolation Polynomial

While choosing the order of the polynomial in a polynomial-type interpolation function, the following considerations have to be taken into account:

- The interpolation polynomial should satisfy, as far as possible, the convergence requirements.
- The pattern of variation of the field variable resulting from the polynomial model should be independent of the local coordinate system.
- The number of generalized coordinates (α_i) should be equal to the number of nodal degrees of freedom of the element (φ_i).
- According to second consideration, it is undesirable to have a preferential coordinate direction i.e. the field variable representation within an element and hence the polynomial should not change with a change in the local coordinate system.
- This property is called geometric isotropy or geometric invariance or spatial isotropy.

Selection of the Order of the Interpolation Polynomial

- In order to achieve geometric isotropy, the polynomial should contain terms that do not violate symmetry which is called **Pascal's triangle** in case of two dimensions and **Pascal tetrahedron** or **pyramid** in the case of three dimensions.



In three dimensions (Pascal tetrahedron or pyramid)

Selection of the Order of the Interpolation Polynomial

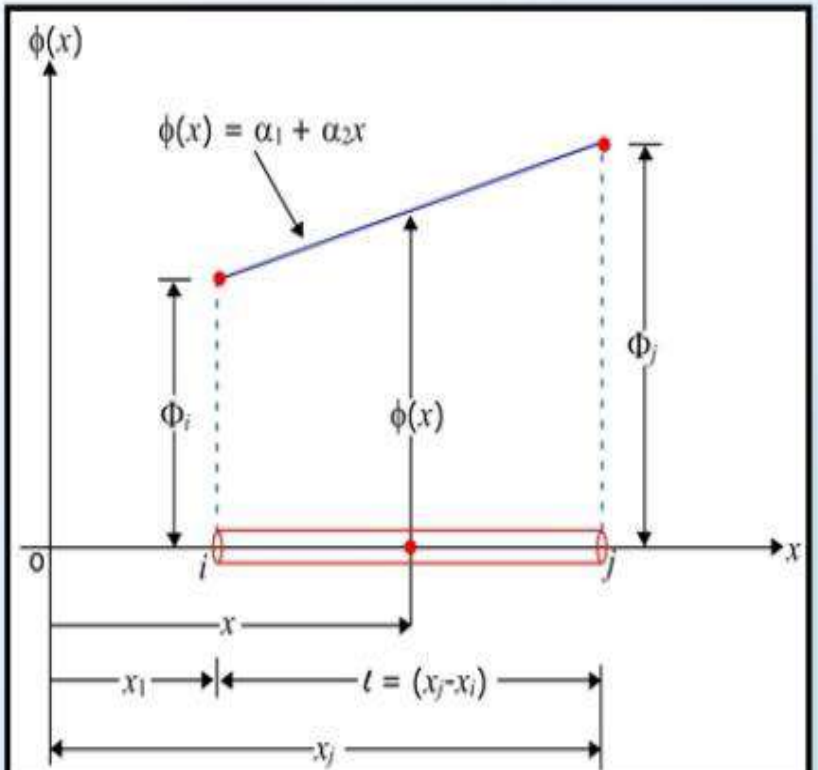
- Thus, in the case of a two-dimensional simplex element (triangle), the interpolation polynomial should include terms containing both x and y, but not only one of them, in addition to the constant term.
- In the case of a two-dimensional complex element (triangle), if we neglect the term x^3 (or x^2y) for any reason, we should not include y^3 (or xy^2) also in order to maintain geometric isotropy of the model.
- Similarly, in the case of a three dimensional simplex element (tetrahedron), the approximating polynomial should contain terms involving x, y, and z in addition to the constant term.
- The final consideration in selecting the order of the interpolation polynomial is to make the total number of terms involved in the polynomial equal to the number of nodal degrees of freedom of the element.

Linear Interpolation of Polynomials in Terms of Global Coordinates

- The linear interpolation polynomials correspond to simplex elements.
- In this section, we will derive the linear interpolation polynomials for the basic one-, two-, and three-dimensional elements in terms of the global coordinates that are defined for the entire domain or body.

One-dimensional simplex element

- Consider a one-dimensional element (line segment) of length l with two nodes, one at each end, as shown.
- Let the nodes be denoted as i and j and the nodal values of the field variable φ as φ_i and φ_j.



One-dimensional simplex element

Linear Interpolation of Polynomials in Terms of Global Coordinates

- The variation of φ inside the element is assumed to be linear as

$$\phi(x) = \alpha_1 x + \alpha_2 x \qquad \text{- - -(21)}$$

where, α_1 and α_2 are the unknown coefficients. By using the nodal conditions

$$\phi(x) = \Phi_i \;\; at \;\; x = x_i$$

$$\phi(x) = \Phi_j \;\; at \;\; x = x_j$$

and equation (21) we get,

$$\Phi_i = \alpha_1 + \alpha_2 x_i$$

$$\Phi_j = \alpha_1 + \alpha_2 x_j$$

- The solution of these equations gives

$$\left.\begin{aligned} \alpha_1 &= \frac{\Phi_i x_j - \Phi_j x_i}{l} \\ \alpha_2 &= \frac{\Phi_j - \Phi_i}{l} \end{aligned}\right\} \qquad \text{- - -(22)}$$

where, x_i and x_j denote the global coordinates of nodes i and j, respectively.

Linear Interpolation of Polynomials in Terms of Global Coordinates

- Substituting equation (22) in equation (21) we get,

$$\phi(x) = \left(\frac{\Phi_i x_j - \Phi_j x_i}{l}\right) + \left(\frac{\Phi_j - \Phi_i}{l}\right) x \qquad \text{- - - (23)}$$

- The equation (23) can be written as

$$\phi(x) = N_i(x)\Phi_i + N_j(x)\Phi_j = \left[N(x)\right]\overrightarrow{\Phi}^{(e)} \qquad \text{- - - (24)}$$

where $\left[N(x)\right] = \left[N_i(x) N_j(x)\right]$ - - - (25)

and

$$\left.\begin{aligned} N_i(x) &= \frac{x_j - x}{l} \\ N_j(x) &= \frac{x - x_i}{l} \end{aligned}\right\} \qquad \text{- - - (26)}$$

- The linear functions of x defined in equation (26) are called interpolation or shape functions.

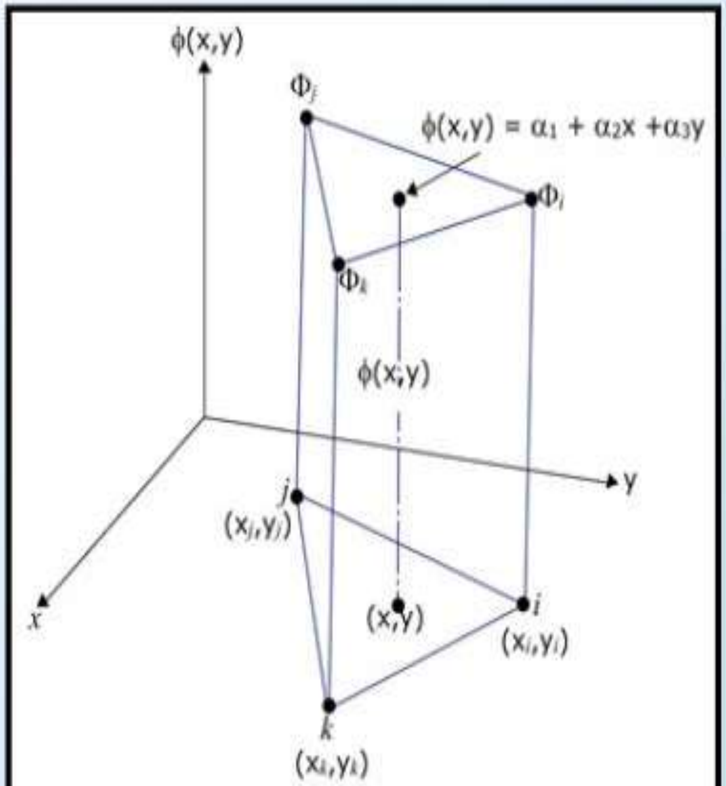
$$\overrightarrow{\Phi}^{(e)} = \begin{Bmatrix} \Phi_i \\ \Phi_j \end{Bmatrix} = \text{vector of nodal unknowns of elements } e \qquad \text{- - - (27)}$$

Linear Interpolation of Polynomials in Terms of Global Coordinates

- The value of $N_i(x)$ can be seen to be 1 at node i $(x = x_i)$ and 0 at node j $(x = x_j)$.
- Likewise, the value of N_i (x) will be 0 at node i and 1 at node j.
- These represent the common characteristics of interpolation functions .
- They will be equal to 1 at one node and 0 at each of the other nodes of the element.

Two-dimensional simplex element

- The two-dimensional simplex element is a straight -sided triangle with three nodes, one at each corner, as indicated.
- Let the nodes be labeled as i, j, and k by proceeding counterclockwise from node i, which is arbitrarily specified



Two-dimensional simplex element

Linear Interpolation of Polynomials in Terms of Global Coordinates

- Let the global coordinates of the nodes i, j and k be given by (x_i, y_i), (x_j, y_j) and (x_k, y_k) and the nodal values of the field variable $\varphi(x, y)$ by $\varphi_i, \varphi_j, \varphi_k$ respectively.
- The variation of φ inside the element is assumed to be linear as

$$\phi(x, y) = \alpha_1 + \alpha_2 + \alpha_3 y \qquad \text{---(28)}$$

- The nodal conditions

$$\phi(x, y) = \Phi_i \quad \text{at} \quad (x = x_i, y = y_i)$$
$$\phi(x, y) = \Phi_j \quad \text{at} \quad (x = x_j, y = y_j)$$
$$\phi(x, y) = \Phi_k \quad \text{at} \quad (x = x_k, y = y_k)$$

Lead to the system of equations

$$\Phi_i = \alpha_1 + \alpha_2 x_i + \alpha_3 y_i$$
$$\Phi_j = \alpha_1 + \alpha_2 x_j + \alpha_3 y_j \qquad \text{---(29)}$$
$$\Phi_k = \alpha_1 + \alpha_2 x_k + \alpha_3 y_k$$

Linear Interpolation of Polynomials in Terms of Global Coordinates

- The solution of equations (29) yields

$$\alpha_1 = \frac{1}{2A}(a_i\Phi_i + a_j\Phi_j + a_k\Phi_k)$$

$$\alpha_2 = \frac{1}{2A}(b_i\Phi_i + b_j\Phi_j + b_k\Phi_k) \qquad \text{---(30)}$$

$$\alpha_3 = \frac{1}{2A}(c_i\Phi_i + c_j\Phi_j + c_k\Phi_k)$$

- The area of the triangle ijk is given by

$$A = \frac{1}{2}\begin{bmatrix} 1 & x_i & y_i \\ 1 & x_j & y_j \\ 1 & x_k & y_k \end{bmatrix} = \frac{1}{2}(x_i y_j + x_j y_k + x_k y_i - x_i y_k - x_j y_i - x_k y_j) \qquad \text{---(31)}$$

$$a_1 = x_j y_k - x_k y_i \;\; b_i = y_j - y_k \;\; c_i = x_k - x_j$$

$$a_j = x_k y_i - x_i y_k \;\; b_j = y_k - y_i \;\; c_j = x_i - x_k \qquad \text{---(32)}$$

$$a_k = x_i y_j + x_j y_i \;\; b_k = y_i - y_j \;\; c_k = x_j - x_i$$

Linear Interpolation of Polynomials in Terms of Global Coordinates

- Substituting equation (30) in (28) and rearranging the equations results in

$$\phi(x,y) = N_i(x,y)\Phi_i + N_j(x,y)\Phi_j + N_k(x,y)\Phi_k = [N(x,y)]\overrightarrow{\Phi}^{(e)} \qquad \text{---(33)}$$

where

$$[N(x,y)] = [N_i(x,y) N_j(x,y) N_k(x,y)] \qquad \text{---(34)}$$

$$N_i(x,y) = \frac{1}{2A}(a_i + b_i x + c_i y)$$

$$N_j(x,y) = \frac{1}{2A}(a_j + b_j x + c_j y) \qquad \text{---(35)}$$

$$N_k(x,y) = \frac{1}{2A}(a_k + b_k x + c_k y)$$

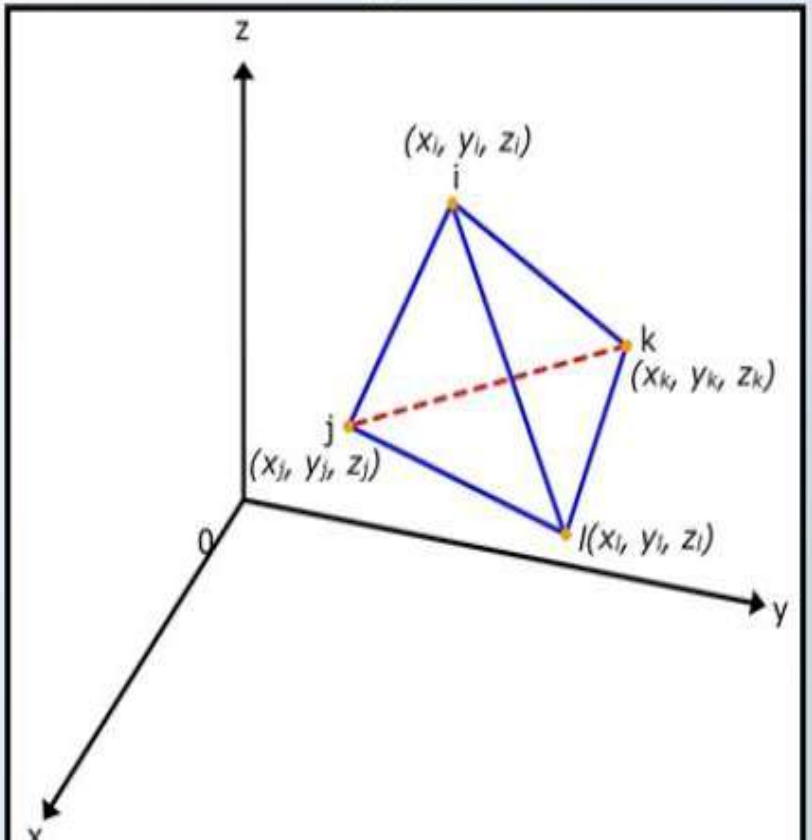
and

$$\overrightarrow{\Phi}^{(e)} = \begin{Bmatrix} \Phi_i \\ \Phi_j \\ \Phi_k \end{Bmatrix} = \textit{vector of nodal unknowns of elements } e \qquad \text{--- (36)}$$

Three-Dimensional Simplex Element

- The three-dimensional simplex element is a flat-faced tetrahedron with four nodes, one at each corner, as shown
- Let the nodes be labeled as i, j, k, and I, where i, j, and k are labeled in a counterclockwise sequence on any face as viewed from the vertex opposite this face, which is labeled as l.
- Let the values of the field variable be φ_i , φ_j , φ_k and φ_l and the global coordinates be (x_i , y_i , z_i), (x_j , y_j , z_j), (x_k , y_k , z_k), and (x_l , y_l , z_l) at nodes i, j, k and l respectively.
- If the variation of φ(x , y , z) is assumed to be linear,

$$\phi(x,y,z)=\alpha_1+\alpha_2 x+\alpha_3 y+\alpha_4 z \qquad \text{---(39)}$$



Three-dimensional simplex element

Three-Dimensional Simplex Element

- The nodal equations $\varphi = \Phi_i$ at (x_i, y_i, z_i), $\varphi = \Phi_j$ at (x_j, y_j, z_j), $\varphi = \Phi_k$ at (x_k, y_k, z_k) and $\varphi = \Phi_l$ *at* (x_l, y_l, z_l) produce the system of equations

$$\begin{aligned}
\Phi_i &= \alpha_1 + \alpha_2 x_i + \alpha_3 y_i + \alpha_4 z_i \\
\Phi_j &= \alpha_1 + \alpha_2 x_j + \alpha_3 y_j + \alpha_4 z_j \\
\Phi_k &= \alpha_1 + \alpha_2 x_k + \alpha_3 y_k + \alpha_4 z_k \\
\Phi_l &= \alpha_1 + \alpha_2 x_l + \alpha_3 y_l + \alpha_4 z_l
\end{aligned} \qquad \text{---(40)}$$

- Equation (40) can be solved and the coefficients α_i, α_j, α_k, α_l are expressed as

$$\begin{aligned}
\alpha_1 &= \frac{1}{6V}(a_i\Phi_i + a_j\Phi_j + a_k\Phi_k + a_l\Phi_l) \\
\alpha_2 &= \frac{1}{6V}(b_i\Phi_i + b_j\Phi_j + b_k\Phi_k + b_l\Phi_l) \\
\alpha_3 &= \frac{1}{6V}(c_i\Phi_i + c_j\Phi_j + c_k\Phi_k + c_l\Phi_l) \\
\alpha_4 &= \frac{1}{6V}(d_i\Phi_i + d_j\Phi_j + d_k\Phi_k + d_l\Phi_l)
\end{aligned} \qquad \text{---(41)}$$

Three-Dimensional Simplex Element

- The volume of the tetrahedron $ijkl$ is given by

$$V = \frac{1}{6}\begin{vmatrix} 1 & x_i & y_i & z_i \\ 1 & x_j & y_j & z_j \\ 1 & x_k & y_k & z_k \\ 1 & x_l & y_l & z_l \end{vmatrix} \qquad \text{---(42)}$$

$$a_i = \begin{vmatrix} x_j & y_j & z_j \\ x_k & y_k & z_k \\ x_l & y_l & z_l \end{vmatrix} \qquad \text{---(43)}$$

$$b_i = -\begin{vmatrix} 1 & y_j & z_j \\ 1 & y_k & z_k \\ 1 & y_l & z_l \end{vmatrix} \qquad \text{---(44)}$$

$$c_i = -\begin{vmatrix} x_j & 1 & z_j \\ x_k & 1 & z_k \\ x_l & 1 & z_l \end{vmatrix} \qquad \text{---(45)}$$

Three-Dimensional Simplex Element

and

$$d_i = -\begin{vmatrix} x_j & y_j & 1 \\ x_k & y_k & 1 \\ x_l & y_l & 1 \end{vmatrix} \quad \text{---(46)}$$

- The signs in front of determinants in equations (43) to (46) are to be reversed when generating a_j, b_j, c_j, d_j and a_l, b_l, c_l, d_l .
- By substituting equations (41) in (39) we get,

$$\phi(x,y,z) = N_i(x,y,z)\phi_i + N_j(x,y,z)\phi_j + N_k(x,y,z)\phi_k + N_l(x,y,z)\phi_l = [N(x,y,z)]\vec{\phi}^{(e)} \quad \text{----(47)}$$

where

$$[N(x,y,z)] = [N_i(x,y,z) + N_j(x,y,z) + N_k(x,y,z) + N_l(x,y,z)]$$

$$N_i(x,y,z) = \frac{1}{6V}(a_i + b_i x + c_i y + d_i z)$$

$$N_j(x,y,z) = \frac{1}{6V}(a_j + b_j x + c_j y + d_j z) \quad \text{---(48)}$$

$$N_k(x,y,z) = \frac{1}{6V}(a_k + b_k x + c_k y + d_k z)$$

Three-Dimensional Simplex Element

$$N_l(x,y,z)=\frac{1}{6V}(a_l+b_lx+c_ly+d_lz)$$

and $$\vec{\phi}^{(e)}=\begin{Bmatrix}\phi_i\\ \phi_j\\ \phi_k\\ \phi_l\end{Bmatrix}$$

C^0 – Continuity:

The one-, two-, and three-dimensional simplex elements considered previously must satisfy the following two properties that imply C^0 -continuity:

- The shape function corresponding to any specific node, such as node i, varies linearly from a value of 1 at that node i to a value of 0 at each of the remaining nodes of the element. Thus, the shape function N_i will have a value of 1 at node i and a value of 0 at each of the remaining nodes of the element.
- The sum of all the shape functions at any point within the element, including its boundaries, will be equal to 1.

Interpolation Polynomials for Vector Quantities

- In equations (21), (28), and (39), the field variable φ has been assumed to be a scalar quantity.
- In some problems the field variable may be a vector quantity having both magnitude and direction.
- In such cases, the usual procedure is to resolve the vector into components parallel to the coordinate axes and treat these components as the unknown quantities.
- Thus, there will be more than one unknown (degree of freedom) at a node in such problems.
- The number of degrees of freedom at a node will be one, two, or three depending on whether the problem is one-, two-, or three-dimensional.

Interpolation Polynomials for Vector Quantities

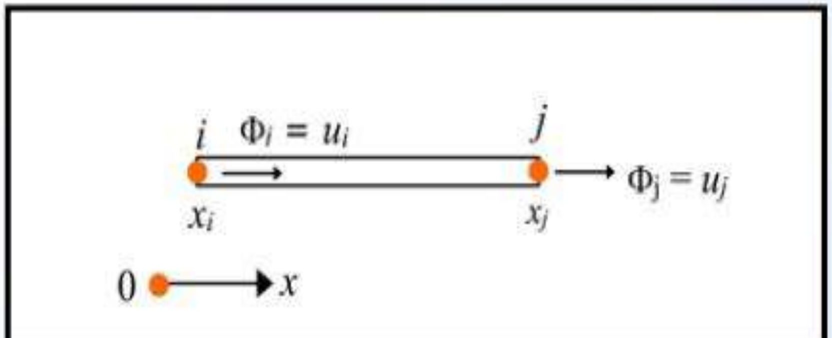
- The interpolation function for a vector quantity in a one dimensional element will be same as that of a scalar quantity since there is only one unknown at each node. Thus,

$$u(x) = N_i(x)\phi_i + N_j(x)\phi_j = [N(x)]\vec{\phi}^{(e)} \qquad \text{---(50)}$$

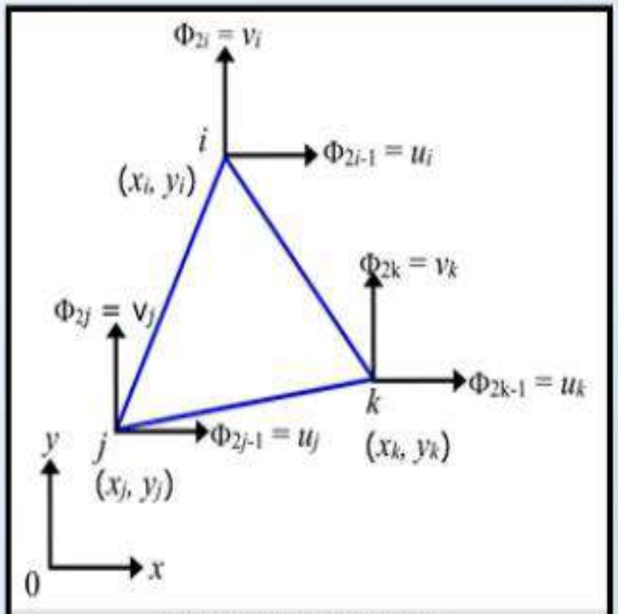
Where, $[N(x)] = [N_i(x) N_j(x)]$

$$\vec{\phi}^{(e)} = \begin{Bmatrix} \phi_i \\ \phi_j \end{Bmatrix}$$

and u is the component of φ parallel to the axis of the element that is assumed to coincide with the x axis.



One-dimensional problem



Two-dimensional problem

Interpolation Polynomials for Vector Quantities

- The shape functions $N_i\,(x)$ and $N_j\,(x)$ are the same as those given in equation (2)
- For a two-dimensional triangular (simplex) element the linear interpolation model of equation (33) will be valid for each of the components of φ, namely, u and v. Thus,

$$u(x,y) = N_i(x,y)\phi_{2i-1} + N_j(x,y)\phi_{2j-1} + N_k(x,y)\phi_{2k-1} \qquad \text{---(51)}$$

$$v(x,y) = N_i(x,y)\phi_{2i} + N_j(x,y)\phi_{2j} + N_k(x,y)\phi_{2k} \qquad \text{---(52)}$$

where ϕ_{2i-1}, ϕ_{2j-1} and ϕ_{2k-1} are nodal values of u ϕ_{2i}, ϕ_{2j} and ϕ_{2k} are the nodal values of v.

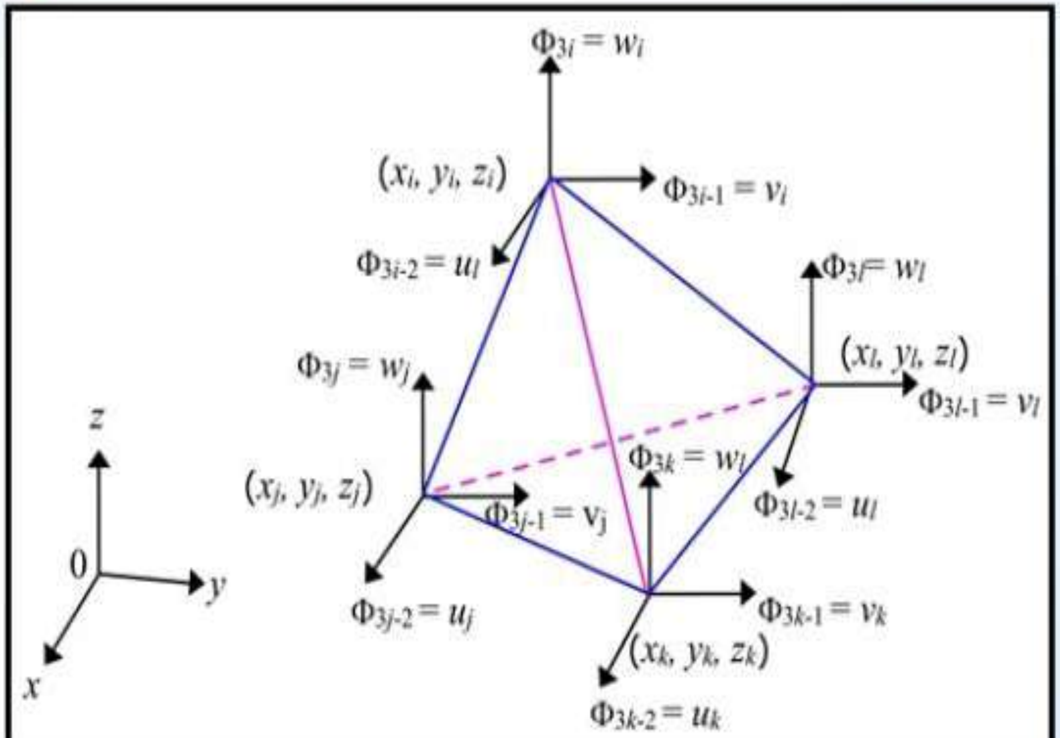
- Equations (51) and (52) can be written in matrix form as

$$\vec{\phi}(x,y) = \begin{Bmatrix} u(x,y) \\ v(x,y) \end{Bmatrix} = [N(x,y)]\vec{\phi}^{(e)} \qquad \text{---(53)}$$

where

$$[\mathrm{N}(x,y)] = \begin{bmatrix} \mathrm{N}_i(x,y) & 0 & \mathrm{N}_j(x,y) & 0 & \mathrm{N}_k(x,y) & 0 \\ 0 & \mathrm{N}_i(x,y) & 0 & \mathrm{N}_j(x,y) & 0 & \mathrm{N}_k(x,y) \end{bmatrix} \qquad \text{---(54)}$$

Interpolation Polynomials for Vector Quantities



Three-dimensional problem

and

$$\vec{\phi}^{(e)} = \begin{Bmatrix} \phi_{2i} - 1 \\ \phi_{2i} \\ \phi_{2j} - 1 \\ \phi_{2j} \\ \phi_{2k} - 1 \\ \phi_{2k} \end{Bmatrix} = \text{vector of nodal degrees of freedom} \qquad ---(55)$$

Interpolation Polynomials for Vector Quantities

- Extending this to three-dimensions, we obtain for a tetrahedron element

$$\vec{\phi}(x,y,z)=\begin{Bmatrix} u(x,y,z) \\ v(x,y,z) \\ w(x,y,z) \end{Bmatrix}=[\mathrm{N}(x,y,z)]\vec{\phi}^{(e)} \qquad \text{---(56)}$$

where

$$[\mathrm{N}(x,y,z)]\begin{bmatrix} N_i(x,y,z) & 0 & 0 & N_j(x,y,z) \\ 0 & N_i(x,y,z) & 0 & 0 \\ 0 & 0 & N_i(x,y,z) & 0 \\ 0 & 0 & N_k(x,y,z) & 0 \\ N_j(x,y,z) & 0 & 0 & N_k(x,y,z) \\ 0 & N_j(x,y,z) & 0 & 0 \\ 0 & N_l(x,y,z) & 0 & 0 \\ 0 & 0 & N_l(x,y,z) & 0 \\ N_k(x,y,z) & 0 & 0 & N_l(x\ y\ z) \end{bmatrix} \qquad \text{---(57)}$$

Interpolation Polynomials for Vector Quantities

$$\vec{\phi}^{(e)} = \begin{bmatrix} \phi_{3i} - 2 \\ \phi_{3i} - 1 \\ \phi_{3i} \\ \phi_{3j} - 2 \\ \phi_{3j} - 1 \\ \phi_{3j} \\ \phi_{3k} - 2 \\ \phi_{3k} - 1 \\ \phi_{3k} \\ \phi_{3l} - 2 \\ \phi_{3l} - 1 \\ \phi_{3i} - 2 \end{bmatrix} \qquad \text{---(58)}$$

Treatment
of
Boundary Conditions

Learning Objectives

At the end of this topic, you will be able to:

- ❖ Derive the linear interpolation polynomials for the basic one-, two-, and three-dimensional elements in terms of the local coordinates.
- ❖ List the convergence requirements for obtaining exact solution.

Outcomes

By the end of this topic, you will be able to:

- Understand the linear interpolation polynomials for the basic one-, two-, and three-dimensional elements in terms of the local coordinates.
- Explain the convergence requirements for obtaining exact solution.

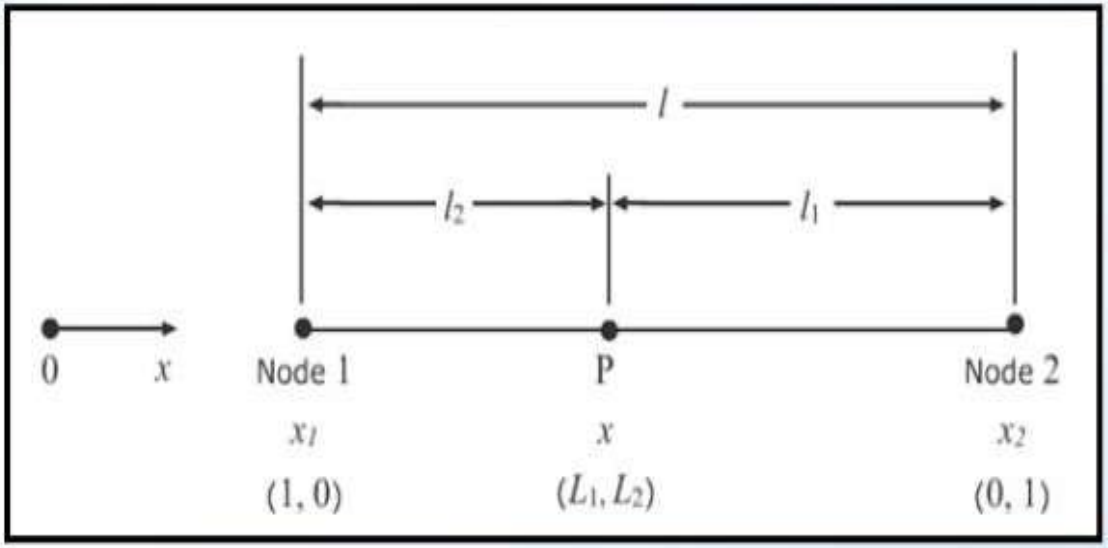
Linear Interpolation Polynomials in Terms of Local Coordinates

- Now, we will derive the interpolation functions of simplex elements in terms of a particular type of local coordinate systems known as natural coordinate systems.
- A natural coordinate system is a local coordinate system that permits the specification of any point inside the element by a set of non dimensional numbers whose magnitude lies between 0 and 1.
- Usually natural coordinate systems are chosen such that some of the natural coordinates will have unit magnitude at primary or corner nodes of the element.

Linear Interpolation Polynomials in Terms of Local Coordinates

One dimensional element

- The natural coordinates of the one dimensional element are as shown.



- Any point P inside the element is identified by two natural coordinates, L_1 and L_2, which are defined as

$$L_1 = \frac{l_1}{l} = \frac{x_2 - x}{x_2 - x_1}$$

$$L_2 = \frac{l_2}{l} = \frac{x - x_1}{x_2 - x_1} \qquad \text{- - - (59)}$$

where,

l_1 and l_2 are the distances shown in image and l is the length of the element.

Linear Interpolation Polynomials in Terms of Local Coordinates

- Since it is a one-dimensional element, there should be only one independent coordinate to define any point P. This is true even with natural coordinates because the two natural coordinates L_1 and L_2 are not independent but are related as

$$L_1 + L_2 = \frac{l_1}{l} + \frac{l_2}{l} = 1 \qquad \text{- - - (60)}$$

- The natural coordinates L_1 and L_2 are also the shape functions for the line element. Comparing the equation (59) with (26), we get

$$N_i = L_1,\ N_j = L_2 \qquad \text{- - - (61)}$$

- Any point x within the element can be expressed as a linear combination of the nodal coordinates of nodes 1 and 2 as

$$x = x_1 L_1 + x_2 L_2 \qquad \text{- - - (62)}$$

where L_1 and L_2 may be interpreted as weighting functions.

Linear Interpolation Polynomials in Terms of Local Coordinates

- The relationship between the natural coordinates and Cartesian coordinates of any point P can be written in matrix form as

$$\begin{Bmatrix} 1 \\ \hline x \end{Bmatrix} = \begin{bmatrix} 1 & 1 \\ x_1 & x_2 \end{bmatrix} \begin{Bmatrix} L_1 \\ \hline L_2 \end{Bmatrix} \qquad \text{- - -(63)}$$

or

$$\begin{Bmatrix} L_1 \\ \hline L_2 \end{Bmatrix} = \frac{1}{(x_2 - x_1)} \begin{bmatrix} x_2 & -1 \\ -x_1 & 1 \end{bmatrix} \begin{Bmatrix} 1 \\ \hline x \end{Bmatrix} = \frac{1}{l} \begin{Bmatrix} x_2 & -1 \\ -x_1 & 1 \end{Bmatrix} \begin{Bmatrix} 1 \\ x \end{Bmatrix} \qquad \text{- - - (64)}$$

- If f is a function of L_1 and L_2 , differentiation of f with respect to x can be performed, using the chain rule, as

$$\frac{df}{dx} = \frac{\partial f}{\partial L_1} \frac{\partial L_1}{\partial x} + \frac{\partial f}{\partial L_2} \frac{\partial L_2}{\partial x} \qquad \text{- - -(65)}$$

Linear Interpolation Polynomials in Terms of Local Coordinates

- From equation (59),

$$\frac{\partial L_1}{\partial x} = -\frac{1}{x_2 - x_1} \; and \; \frac{\partial L_2}{\partial x} = \frac{1}{x_2 - x_1} \qquad \text{- - -(66)}$$

- Integration of polynomial terms in natural coordinates can be performed by using the simple formula

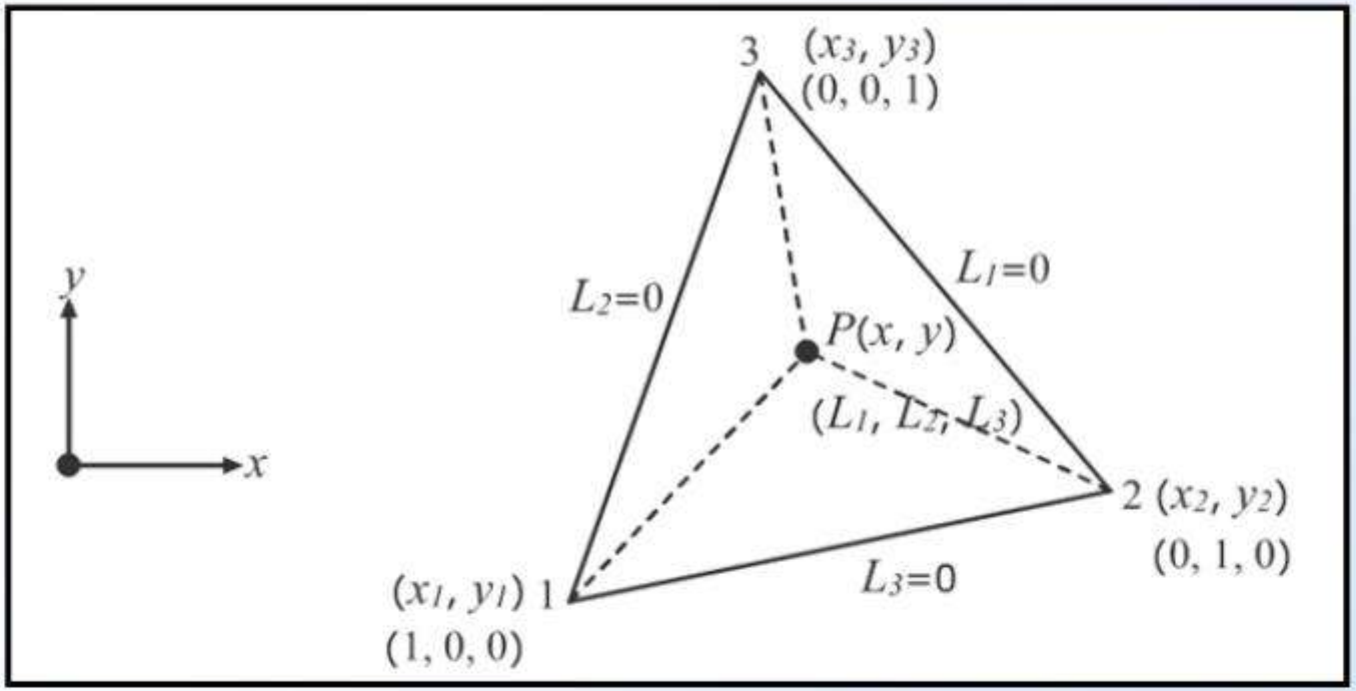
$$\int_{x_1}^{x_2} L_1^{\alpha} L_2^{\beta} = \frac{\alpha!\beta!}{(\alpha+\beta+1)!} l \qquad \text{- - - (67)}$$

where

α_1 is the factorial of α and it is given by $\alpha! = \alpha(\alpha\text{-}1)(\alpha\text{-}2)$........(1)

Two-Dimensional (Triangular) Element

A natural coordinate system for a triangular element is as shown



Two-Dimensional (Triangular) Element

- Although three coordinates L_1 , L_2, and L_3 are used to define a point P, only two of them are independent. The natural coordinates are defined as

$$L_1 = \frac{A_1}{A},\ L_2 = \frac{A_2}{A},\ L_3 = \frac{A_3}{A} \qquad \text{- - - (68)}$$

where A_1 is the area of the triangle formed by the points P, 2 and 3; A_2 is the area of the triangle formed by the points P, 1 and 3; A_3 is the area of the triangle formed by the points P, 1 and 2; and A is the area of the triangle 123.

- Because L_i are defined in terms of areas, they are also known as area coordinates. Since we have

$$A_1 + A_2 + A_3 = A \qquad \text{- - - (69)}$$

also

$$\frac{A_1}{A} + \frac{A_2}{A} + \frac{A_3}{A} = L_1 + L_2 + L_3 = 1 \qquad \text{- - -(70)}$$

$$N_1 = L_1,\ \mathrm{N}_2 = L_2,\ \mathrm{N}_3 = L_3$$

Two-Dimensional (Triangular) Element

- The relation between the natural and Cartesian coordinates is given by

$$x = x_1 L_1 + x_2 L_2 + x_3 L_3 \qquad \text{- - - (71)}$$
$$y = y_1 L_1 + y_2 L_2 + y_3 L_3$$

- Equations (69) and (71) can be written in matrix form as

$$\begin{Bmatrix} 1 \\ x \\ y \end{Bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix} \begin{Bmatrix} L_1 \\ L_2 \\ L_3 \end{Bmatrix} \qquad \text{- - - (72)}$$

- Equation (72) can be inverted to obtain,

$$\begin{Bmatrix} L_1 \\ L_2 \\ L_3 \end{Bmatrix} = \frac{1}{2A} \begin{bmatrix} (x_2 y_3 - x_3 y_2) & (y_2 - y_3) & (x_3 - x_2) \\ (x_3 y_1 - x_1 y_3) & (y_3 - y_1) & (x_1 - x_3) \\ (x_1 y_2 - x_2 y_1) & (y_1 - y_2) & (x_2 - x_1) \end{bmatrix} \begin{Bmatrix} 1 \\ x \\ y \end{Bmatrix} \qquad \text{- - - (73)}$$

Two-Dimensional (Triangular) Element

- The area of the triangle A is given by

$$A = \frac{1}{2}\begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix} \qquad \text{- - -(74)}$$

- If f is a function of L_1, L_2, and L_3, the differentiation with respect to x and y can be performed as

$$\frac{\partial f}{\partial x} = \sum_{i=1}^{3} \frac{\partial f}{\partial L_i}\frac{\partial L_i}{\partial x}$$
$$\frac{\partial f}{\partial y} = \sum_{i=1}^{3} \frac{\partial f}{\partial L_i}\frac{\partial L_i}{\partial y} \qquad \text{- - -(75)}$$

where

$$\frac{\partial L_1}{\partial x} = \frac{y_2 - y_3}{2A}, \quad \frac{\partial L_1}{\partial y} = \frac{x_3 - x_2}{2A}$$
$$\frac{\partial L_2}{\partial x} = \frac{y_3 - y_1}{2A}, \quad \frac{\partial L_3}{\partial y} = \frac{x_1 - x_3}{2A} \qquad \text{- - - (76)}$$
$$\frac{\partial L_3}{\partial x} = \frac{y_1 - y_2}{2A}, \quad \frac{\partial L_3}{\partial y} = \frac{x_2 - x_1}{2A}$$

Two-Dimensional (Triangular) Element

- For integrating polynomial terms in natural coordinates, we can use the relations

$$\int_L L_1^\alpha L_2^\beta . dL = \frac{\alpha!\beta!}{(\alpha+\beta+1)!} L \qquad \text{- - - (77)}$$

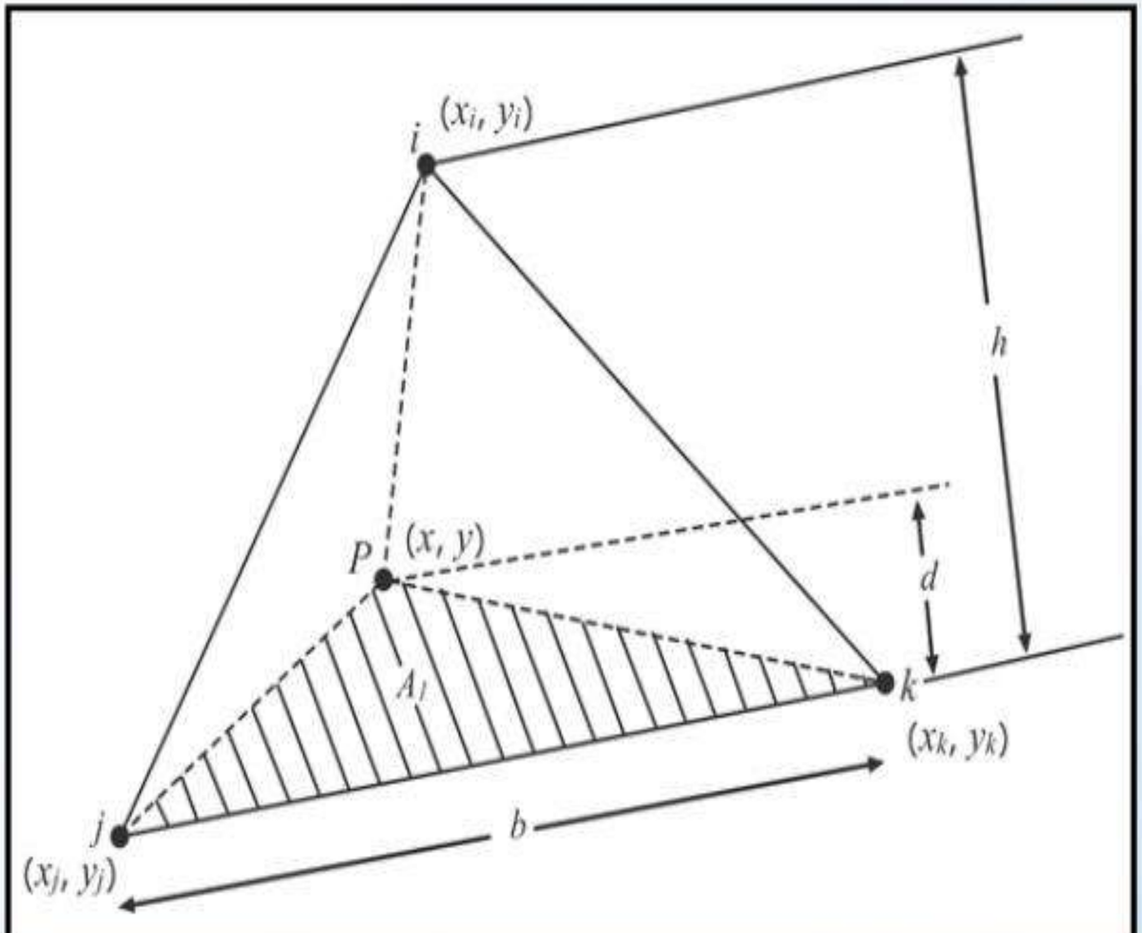
and

$$\iint_A L_1^\alpha L_2^\beta L_3^\gamma . dA = \frac{\alpha!\beta!\gamma!}{(\alpha+\beta+\gamma+2)!} 2A \qquad \text{- - - (78)}$$

- Equation (77) is used to evaluate an integral that is a function of the length along an edge of the element.
- Equation (78) is used to evaluate area integrals.

Example

Show that the natural (area) coordinate Li $(i = 1, 2, 3)$ is the same as the shape function Ni given by Eq. (35).



Example

Solution

The area coordinate LI, defined as the ratio of the area of the shaded triangle to the total area of the triangle ijk shown in Figure can be expressed as

$$L_1 = \frac{A_1}{A} = \frac{\frac{1}{2}bd}{\frac{1}{2}bh} = \frac{d}{h} \qquad \text{---(E.1)}$$

where d and h denote the distances of the perpendiculars from the points P and i to the base jk of the triangle.

Example

- The area AI of the triangle Pjk can be determined in terms of the coordinates of $P, j,$ and k as

$$2A_1 = \begin{vmatrix} 1 & x & y \\ 1 & x_j & y_j \\ 1 & x_k & y_k \end{vmatrix} = x_j y_k - x_k y_j + x(y_j - y_k) + y(x_k - x_j) \qquad \text{(E.2)}$$

Equations (E. I) and (E.2) lead to

$$L_1 = \frac{2A_1}{2A} = \frac{1}{2A}(x_j y_k) + x(y_j - y_k) + y(x_k - x_j) \qquad \text{(E.3)}$$

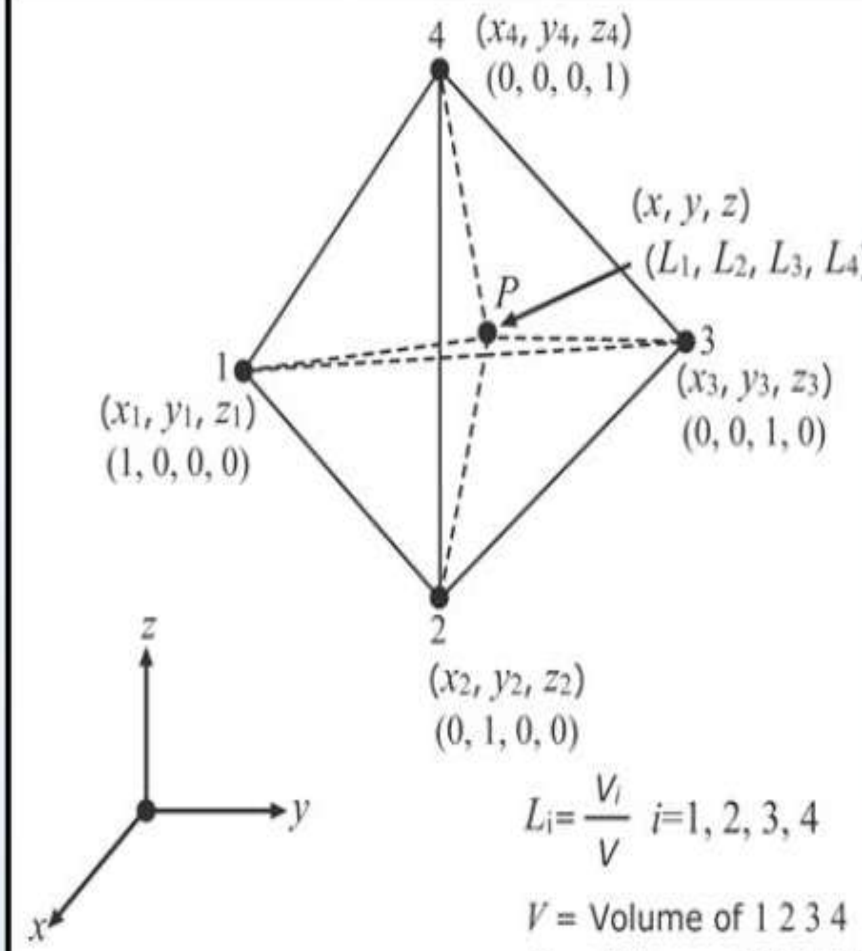
which can be seen to be identical to the shape function Ni given by Eq. (35).

Three-Dimensional (Tetrahedron) Element

- The natural coordinates for a tetrahedron element can be defined analogous to those of a triangular element.

  - Thus, four coordinates L_1, L_2, L_3, and L_4 will be used to define a point P, although only three of them are independent.
  - These natural coordinates are defined as

$$L_1 = \frac{V_1}{V},\ L_2 = \frac{V_2}{V},\ L_3 = \frac{V_3}{V},\ L_4 = \frac{V_4}{V} \quad ---(79)$$



$L_i = \frac{V_i}{V}$ $i=1, 2, 3, 4$

V = Volume of 1 2 3 4

V_1 = Volume of P 2 3 4

V_2 = Volume of P 1 3 4

V_3 = Volume of P 1 2 4

V_4 = Volume of P 1 2 3

Three-Dimensional (Tetrahedron) Element

- V_I is the volume of the tetrahedron formed by the points P and the vertices other than the vertex i $(i = 1, 2, 3, 4)$, and V is the volume of the tetrahedron element defined by the vertices 1,2,3, and 4.
- Because the natural coordinates are defined in terms of volumes. they are also known as volume or tetrahedral coordinates. Since

$$V_1 + V_2 + V_3 + V_4 = V$$

we get

$$\frac{V_1}{V} + \frac{V_2}{V} + \frac{V_3}{V} + \frac{V_4}{V} = L_1 + L_2 + L_3 + L_4 = 1 \qquad \text{- - -(80)}$$

- The volume coordinates L_1 , L_2, L_3 and L_4 are also the shape functions for a three-dimensional simplex element:

$$N_i = L_1,\ N_j = L_2,\ N_k = L_3,\ N_l = L_4 \qquad \text{- - -(81)}$$

Three-Dimensional (Tetrahedron) Element

- The Cartesian and natural coordinates are related as

$$\begin{aligned} x &= L_1x_1 + L_2x_2 + L_3x_3 + L_4x_4 \\ y &= L_1y_1 + L_2y_2 + L_3y_3 + L_4y_4 \\ z &= L_1z_1 + L_2z_2 + L_3z_3 + L_4z_4 \end{aligned} \quad \text{- - - (82)}$$

- Equations (80) and (82) can be expressed in matrix form as

$$\begin{Bmatrix} 1 \\ x \\ y \\ z \end{Bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ z_1 & z_2 & z_3 & z_4 \end{bmatrix} \begin{Bmatrix} L_1 \\ L_2 \\ L_3 \\ L_4 \end{Bmatrix} \quad \text{- - - (83)}$$

- The inverse relations can be expressed as

$$\begin{Bmatrix} L_1 \\ L_2 \\ L_3 \\ L_4 \end{Bmatrix} = \frac{1}{6V} \begin{bmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{bmatrix} \begin{Bmatrix} 1 \\ x \\ y \\ z \end{Bmatrix} \quad \text{- - -(84)}$$

Three-Dimensional (Tetrahedron) Element

where

$$V=\frac{1}{6}\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix} = volume\ of\ the\ tetrahedon\ 1,2,3,4 \qquad ---(85)$$

$$a_1=\begin{vmatrix} x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \\ x_4 & y_4 & z_4 \end{vmatrix} \qquad ---(86)$$

$$b_1=-\begin{vmatrix} 1 & y_2 & z_2 \\ 1 & y_3 & z_3 \\ 1 & y_4 & z_4 \end{vmatrix} \qquad ---(87)$$

$$c_1=-\begin{vmatrix} x_2 & 1 & z_2 \\ x_3 & 1 & z_3 \\ x_4 & 1 & z_4 \end{vmatrix} \qquad ---(88)$$

Three-Dimensional (Tetrahedron) Element

$$d_1 = \begin{vmatrix} x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \\ x_4 & y_4 & 1 \end{vmatrix} \qquad \text{- - -(89)}$$

- These constants are the co factors of the terms in the determinant of equation (85).
- If f is a function of the natural coordinates, it can be differentiated with respect to Cartesian coordinates as

$$\frac{\partial f}{\partial x} = \sum_{i=1}^{4} \frac{\partial f}{\partial L_i}\frac{\partial L_i}{\partial x}$$

$$\frac{\partial f}{\partial y} = \sum_{i=1}^{4} \frac{\partial f}{\partial L_i}\frac{\partial L_i}{\partial y} \qquad \text{- - - (90)}$$

$$\frac{\partial f}{\partial z} = \sum_{i=1}^{4} \frac{\partial f}{\partial L_i}\frac{\partial L_i}{\partial z}$$

Three-Dimensional (Tetrahedron) Element

where

$$\frac{\partial L_i}{\partial x} = \frac{b_i}{6V}, \quad \frac{\partial L_i}{\partial y} = \frac{c_i}{6V}, \quad \frac{\partial L_i}{\partial z} = \frac{d_i}{6V} \qquad \text{- - -(91)}$$

- The integration of polynomial terms in natural coordinates can be performed using the relation

$$\iiint_v L_1^{\alpha} L_2^{\beta} L_3^{\gamma} L_4^{\delta} dV = \frac{\alpha!\beta!\gamma!\delta!}{(\alpha+\beta+\gamma+\delta+3)!} 6V \qquad \text{- - -(92)}$$

Example

Show that the natural coordinate 1; is same as the shape function N, given by Eq. (48).

Approach: Use the definition of the volume of a tetrahedron.

Solution:

Using the definition of L ; given in Eq. (79), we have

$$L_i = \frac{\text{volume of tetrahedron pjkl}}{\text{volume of tetrahedron ijkl}} = \frac{V_i}{V} \tag{E.1}$$

Where, the volume V, can be expressed as (from Figure):

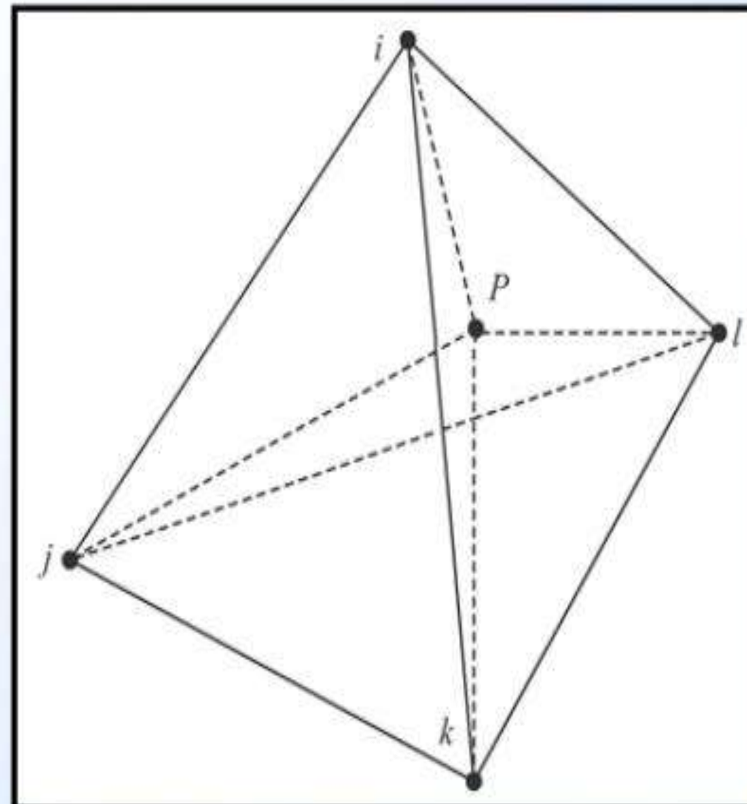
$$V_i = \frac{1}{6}\begin{vmatrix} 1 & x & y & z \\ 1 & x_j & y_j & z_j \\ 1 & x_k & y_k & z_k \\ 1 & x_l & y_l & z_l \end{vmatrix}$$

$$= \frac{1}{6}\left\{1\begin{vmatrix} x_j & y_j & z_j \\ x_k & y_k & z_k \\ x_l & y_l & z_l \end{vmatrix} - x\begin{vmatrix} 1 & y_j & z_j \\ 1 & y_k & z_k \\ 1 & y_l & z_l \end{vmatrix} + y\begin{vmatrix} 1 & x_j & z_j \\ 1 & x_k & z_k \\ 1 & x_l & z_l \end{vmatrix} - z\begin{vmatrix} 1 & x_j & y_j \\ 1 & x_k & y_k \\ 1 & x_l & y_l \end{vmatrix}\right\} \tag{E.2}$$

Example

Using Eqs. (43) to (46), Eq. (E.2) can be rewritten as

$$V_i = \frac{1}{6}(a + b_1 x + c_1 y + d_1 z) \qquad \text{(E.3)}$$



Thus, Li, defined by Eq. (E. I), can be expressed as

$$L_i = \frac{1}{6V}(a + b_1 x + c_1 y + d_1 z) \qquad \text{(E.4)}$$

Which can be seen to be identical to the expression of N, given in Eq. (48).

Convergence Requirements

Since the finite element method is a numerical technique, we obtain a sequence of approximate solutions as the element size is reduced successively. This sequence will converge to the exact solution if the interpolation polynomial satisfies the following convergence requirements.

- The field variable must be continuous within the elements.
- All uniform states of the field variable φ and its partial derivatives up to the highest order appearing in the functional $I(\varphi)$ must have representation in the interpolation polynomial when, in the limit, the element size reduces to zero.
- The field variable φ and its partial derivatives up to one order less than the highest order derivative appearing in the functional $I(\varphi)$ must be continuous at element boundaries or interfaces.

Convergence Requirements

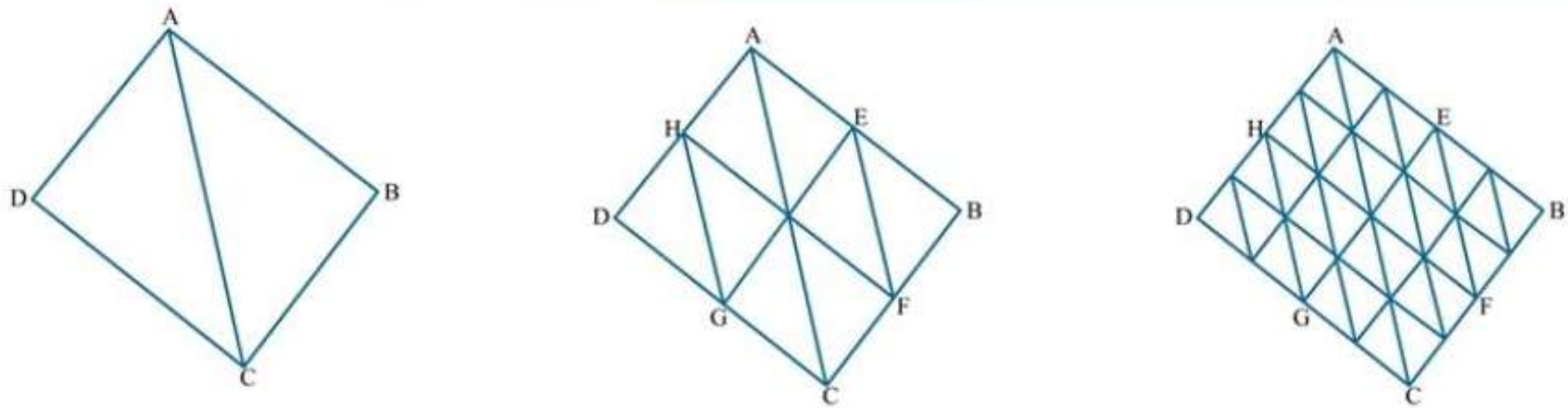
- The elements whose interpolation polynomials satisfy the requirements (1) and (3) are called **compatible or conforming elements** and those satisfying condition (2) are called **complete elements.**

- If r^{th} derivative of the field variable φ is continuous, then φ is said to have C^r continuity. In terms of this notation, the completeness requirement implies that φ must have C^r continuity within an element, whereas the compatibility requirement implies that φ must have C^{r-1} continuity at element interfaces.

- If the interpolation polynomial satisfies all three requirements, the approximate solution converges to the correct solution when we refine the mesh and use an increasing number of smaller elements

Convergence Requirements

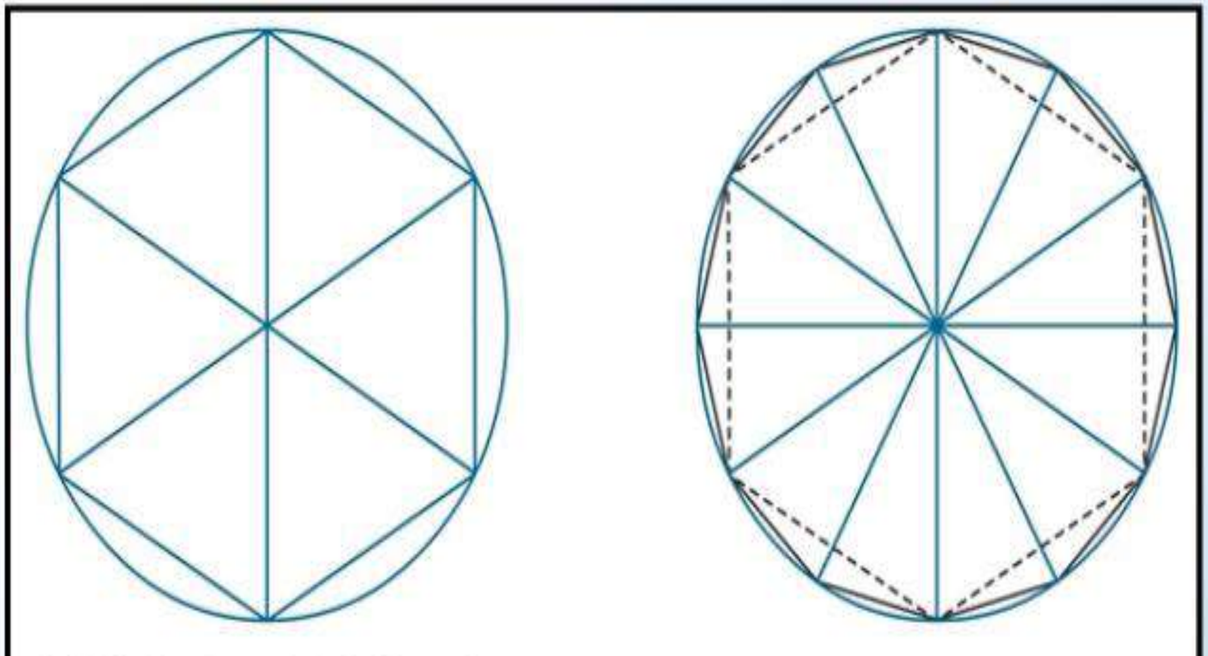
- In order to prove the convergence mathematically, the mesh refinement has to be made in a regular fashion so as to satisfy the following conditions:
  1. All previous (coarse) meshes must be contained in the refined meshes.
  2. The elements must be made smaller in such a way that every point of the solution region can always be within an element.
  3. The form of the interpolation polynomial must remain unchanged during the process of mesh refinement. Conditions 1 and 2 are illustrated as shown.



(a) Idealization with 2 elements (b) Idealization with 8 elements (c) Idealization with 32 elements

Convergence Requirements

- From Figure, in which the solution region is assumed to have a curved boundary, it can be seen that conditions (1) and (2) are not satisfied if we use elements with straight boundaries.
- In structural problems, interpolation polynomials satisfying all the convergence requirements always lead to the convergence of the displacement solution from below while nonconforming elements may converge either from below or from above.



(a) Idealization with 6 elements

(b) Idealization with 12 elements

Treatment of Boundary Conditions

Prescribed temperature (Dirichlet condition)

- Boundary gives a value to the problem (i.e.) Temperature distribution at a boundary surface known.

  Example: One end of an iron rod is held at absolute zero.

Prescribed heat flux (Neumann condition)

- Boundary gives a value to the normal derivative of the problem. **Example:** A heater at one end of an iron rod, then energy would be added at a constant rate but the actual temperature would not be known.

Convective boundary condition (Robin condition)

- Boundary is subjected to convective heat transfer with a fluid at ambient.

Analysis of Trusses

Learning Objectives

At the end of this topic, you will be able to:

- Know Finite element modeling, coordinates and shape functions of a truss
- Understand the finite element Equations and the terms involved
- Explain treatment of boundary conditions involved in truss element
- Calculate stress , strain and support reactions of a truss element

Outcomes

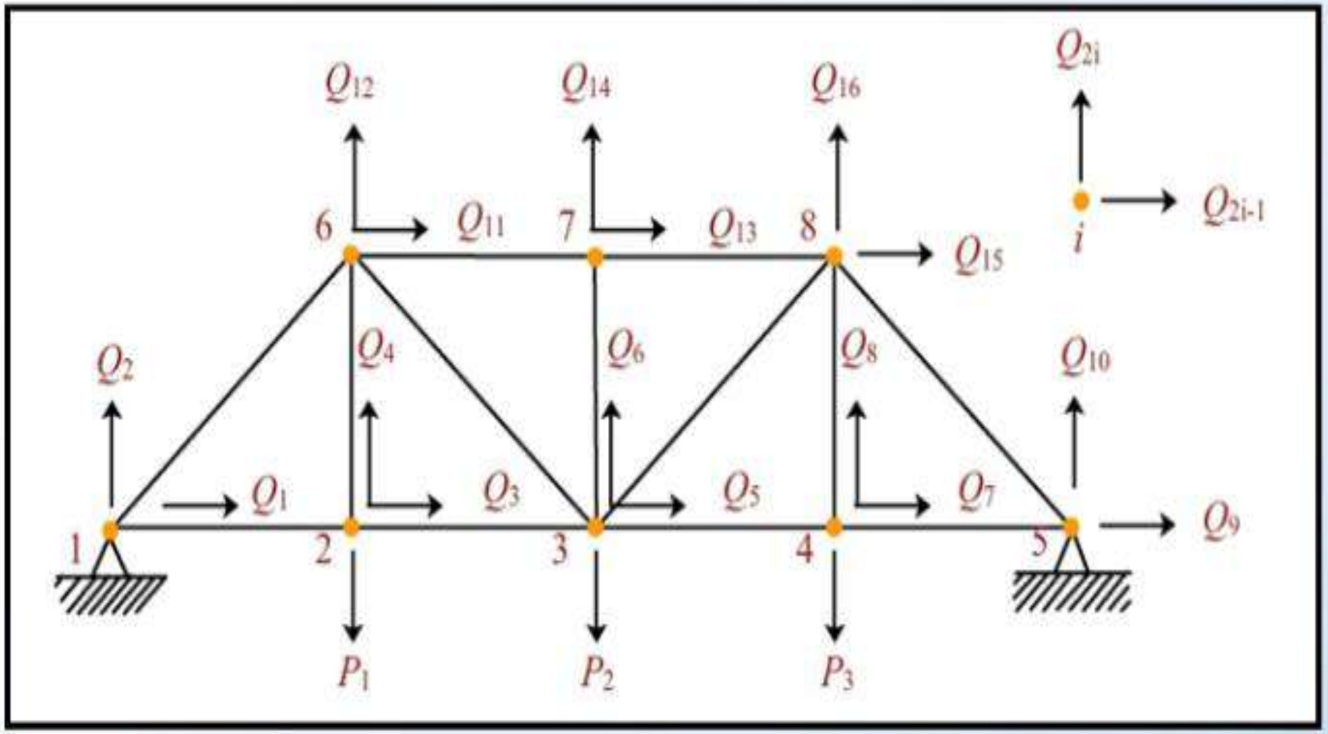
By the end of this topic, you will be able to:

- Understand the Finite element modeling, coordinates and shape functions of a truss
- Explain about the finite element Equations and the terms involved
- Discuss the treatment of boundary conditions involved in truss element
- Work out the stress, strain and support reactions of a truss element

Introduction to Trusses

- A truss structure consists only of two-force members. That is, every truss element is in direct tension or compression.

- In a truss, it is required that all loads and reactions are applied only at the joints and that all members are connected together at their ends by frictionless pin joints.



Example of Two Dimensional Stress

Types of Trusses

Pratt truss

- The design uses vertical members for compression and horizontal members to respond to tension. The continued popularity of the Pratt truss is probably due to the fact that the configuration of the members means that longer diagonal members are only in tension for gravity load effects.

Warren truss

- The warren truss has made up of several bar pin jointed or welded. At the final stage the ends will be either roller supported or pinned supported. It has a good strength

Space truss

- In space truss all members are not in the same plane
- Many space trusses having plane truss. Several plane truss joined together form a space truss Plane truss
- In plane truss all the members are in the same plane
- The forces acting only on the plane of members

Finite Element Modeling of Truss

- Finite element modeling is the procedure of discritization of a structure and the numbering of the element.

Discritization

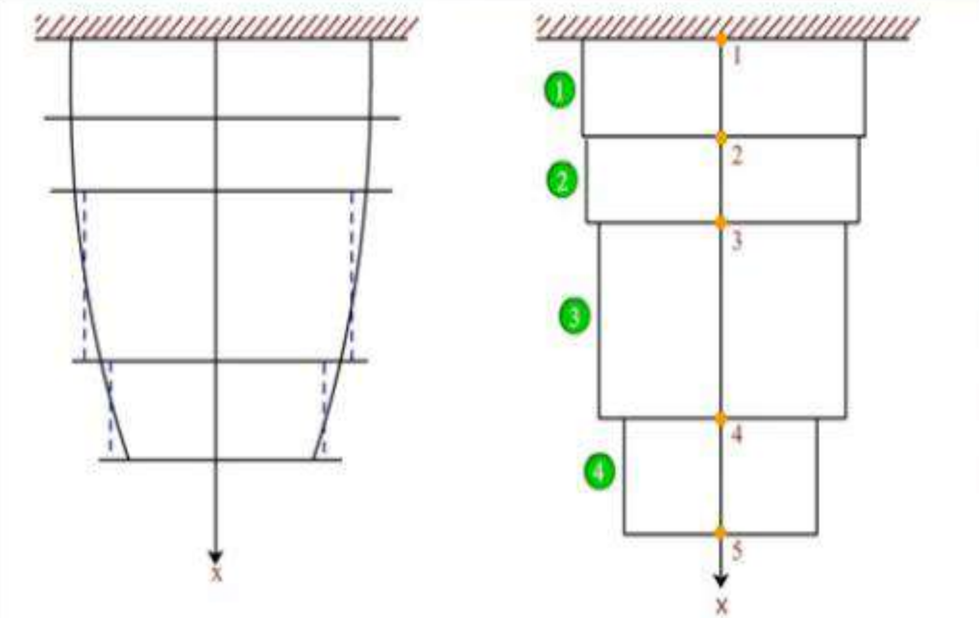
- It is a process of subdividing the whole structure into a small finite elements of equal size and shape for the better accurate solutions of the object.

Numbering of the Element

- After subdividing the whole into small elements the numbering has to be done for creating stiffness matrix and global stiffness matrix.

Discritization of a Structure

- The first step is to model the bar as a stepped shaft, consisting of a discrete number of elements, each having a uniform cross section. Specifically, let us model the bar using four finite elements.
- The average cross-sectional area within each region is evaluated and then used to define an element with uniform cross section. In addition to the cross section, traction and body forces are also (normally) treated as constant within each element
- However, cross-sectional area, traction, and body forces can differ in magnitude from element to element. Better approximations are obtained by increasing the number of elements. It is convenient to define a node at each location where a point load is applied.

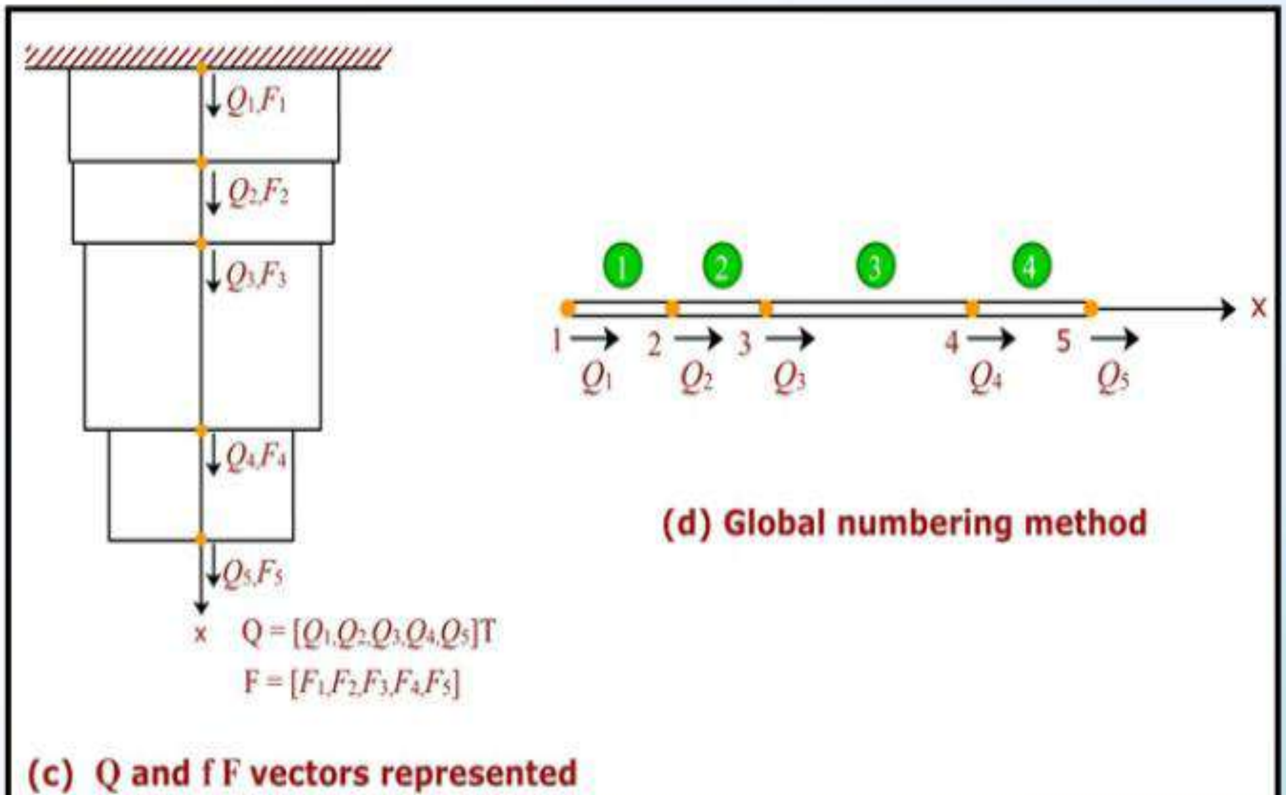


(a) Shows the element before discritization

(b) Shows the element after discretization

Numbering of a Structure

- In a one-dimensional problem, every node is permitted to displace only in the $\pm x$ direction. Thus, each node has only one degree of freedom (dof) . The five-node finite element model is considered which has five dofs.
- The displacements along each dof are denoted by $Q_1, Q_2, \cdots, Q_s$. In fact, the column vector $Q = [Q_1, Q_2, \ldots, Q_S]^T$ is called the global displacement vector.

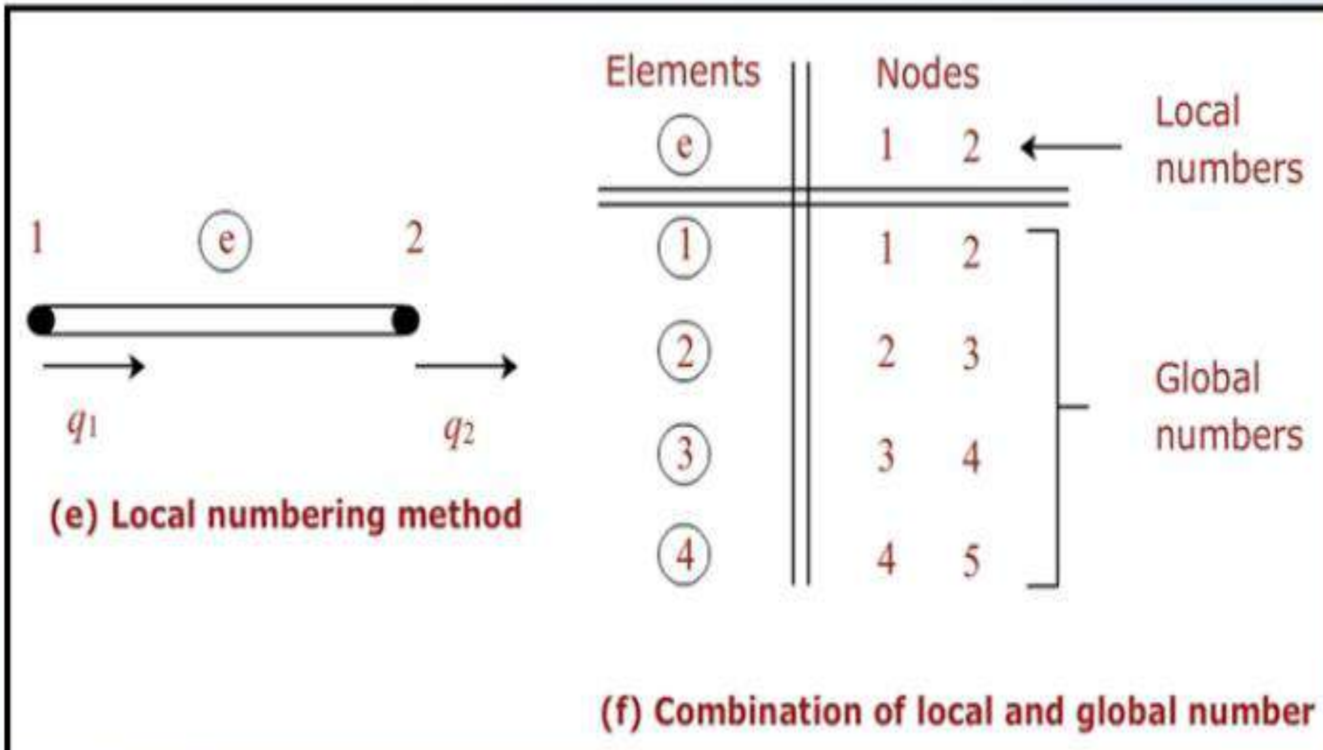


(c) Q and f F vectors represented

(d) Global numbering method

Numbering of a Structure

- The global load vector is denoted by $F = [F_1, F_2, \ldots, F_s]^T$. The sign convention used is that a displacement or load has a positive value if acting along the $+x$ direction. At this stage, conditions at the boundary are not imposed

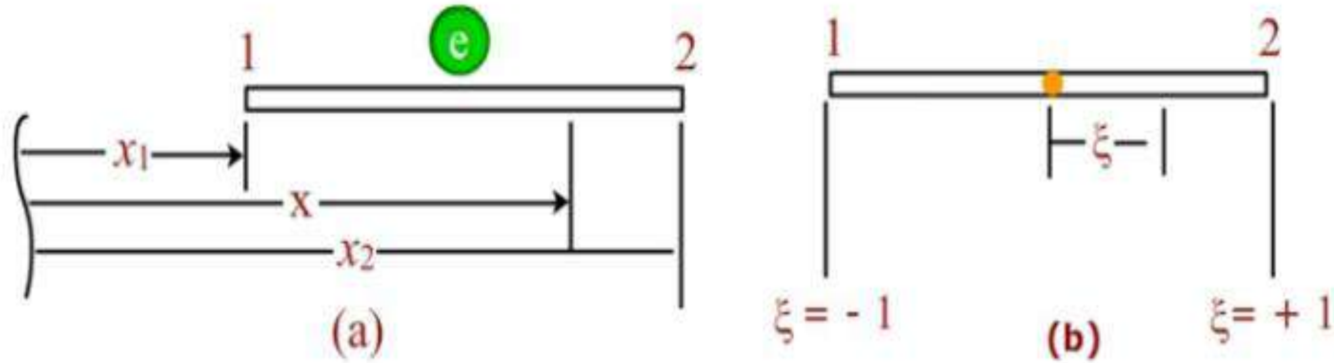


(e) Local numbering method

(f) Combination of local and global number

Coordinates and Shape Functions

In the local number. scheme, the first node will be numbered 1 and the second node 2. The notation X_1 = x-coordinate of node 1, X_2 = x-coordinate of node 2 is used. We define a **natural or intrinsic coordinate** system, as



Typical Elements in x-and ξ-Coordinates

The system of coordinates in defining shape functions, which are used in interpolating the displacement field

$$\xi = \frac{2}{x_2 - x_1}(x - x_1) - 1$$

Coordinates and Shape Functions

The unknown displacement field within an element will be interpolated by a linear distribution. This approximation becomes increasingly accurate as more elements are considered in the model. To implement this linear interpolation, linear shape functions will be introduced as,

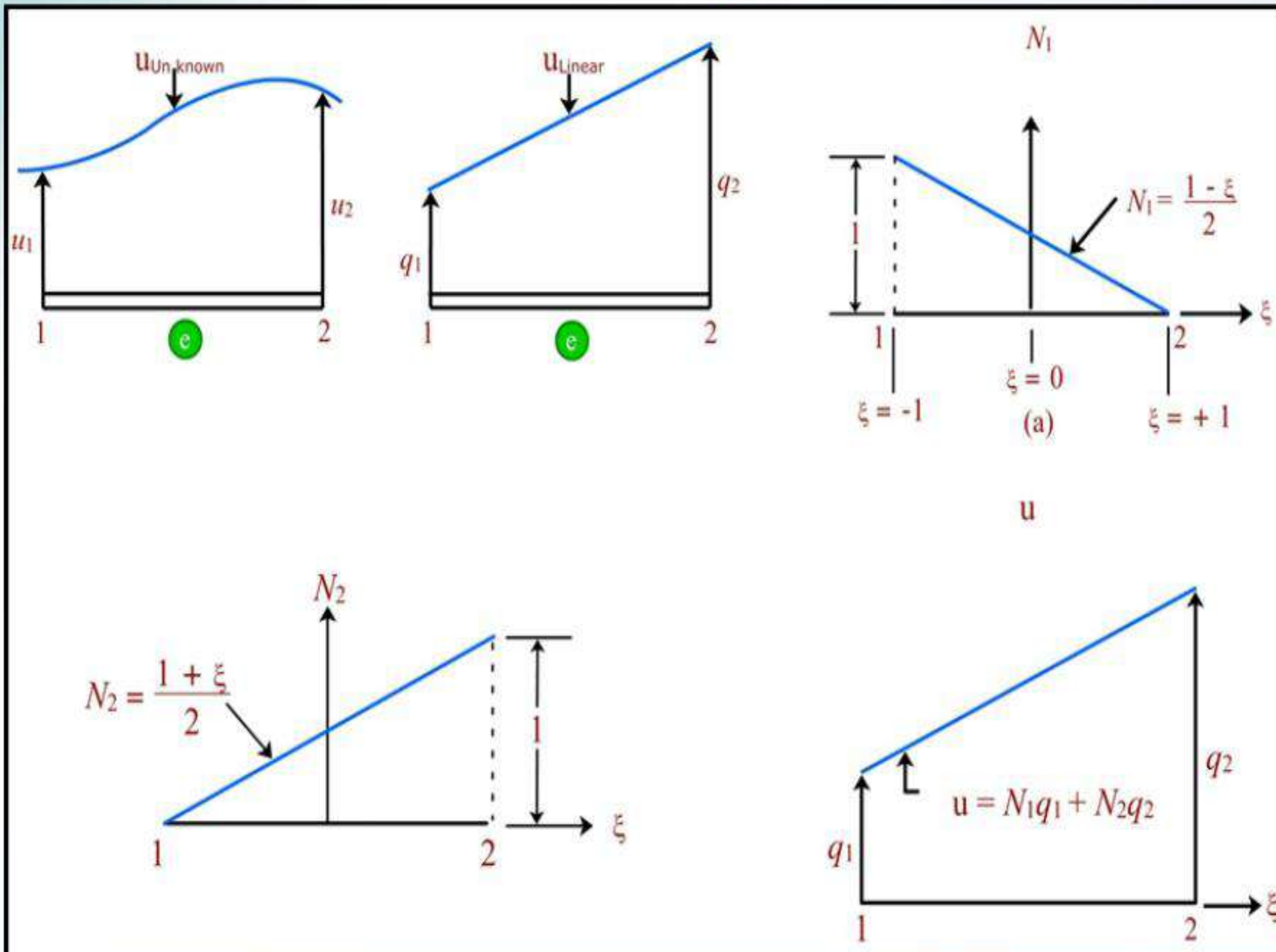
$$u = N_1 q_1 + N_2 q_2$$

Once the shape functions are defined, the linear displacement field within the element can be written in terms of the nodal displacements q_1 and q_2 as,

$$N_1(\xi) = \frac{1-\xi}{2}$$

$$N_2(\xi) = \frac{1+\xi}{2}$$

Coordinates and Shape Functions



Coordinates and Shape Functions

In matrix notation,

$$u = Nq$$

Where,

$$N = [N_1, N_2] \ and \ q = [q_1, q_2]^T$$

In these equations, q is referred to as the element displacement vector. It is readily verified that $u = q_1$ at node 1, $u = q_2$ at node 2, and that u varies linearly.

$$x = N_1 x_1 + N_2 x_2$$

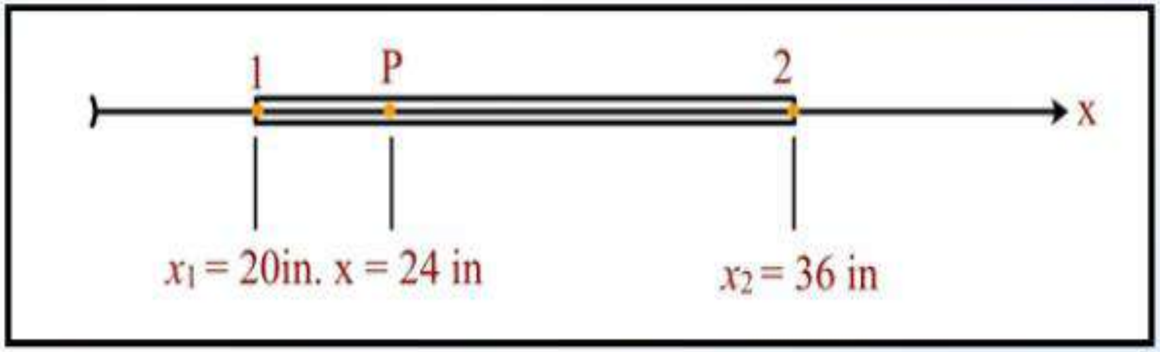
The displacement u and the coordinate x are interpolated within the element using the same shape functions N_1 and N_2. This is referred to as the isoperimetric formulation in the literature.

Properties of the Shape Functions:

1. First derivatives must be finite within an element.
2. Displacements must be continuous across the element boundary.

Problems on Coordinates and Shape Functions

Problem 1:



(a) Evaluate, N_1 , and N_2 at point P.

(b) If $q_1 = 0.003$ in. and q_2 = -0.005 *in.*, determine the value of the displacement q, at point P.

Solution:

when we use the equation below can get shape functions at 1 and 2.

$$\xi = \frac{2}{x_2 - x_1}(x - x_1) - 1$$

$$\xi_p = \frac{2}{16}(24 - 20) - 1$$

$$= -0.5$$

Problems on Coordinates and Shape Functions

To fine the shape function we have,

$$N_1(\xi) = \frac{1-\xi}{2}$$

$$N_2(\xi) = \frac{1+\xi}{2}$$

After substituting the values can get ,

$$N_1 = 0.75 \, , \, N_2 = 0.25$$

Using this equation can fine displacement in each element.

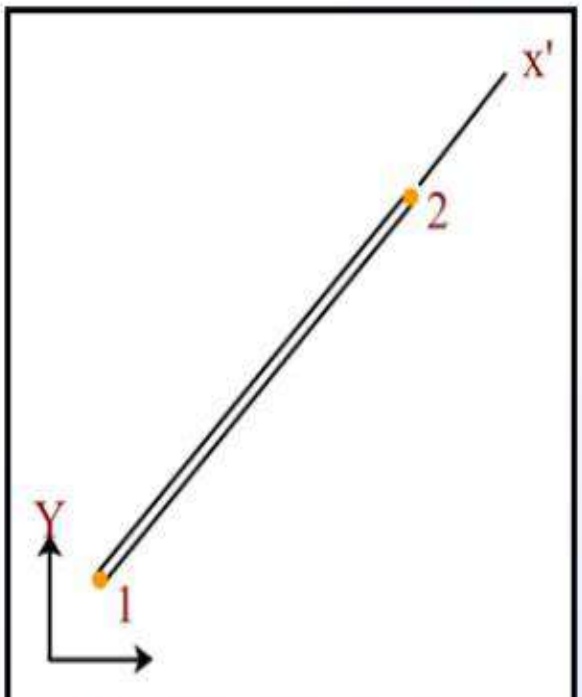
$$u = N_1 q_1 + N_2 q_2$$

$$u = 075 \times 0.003 + 0.25 \times -0.005$$

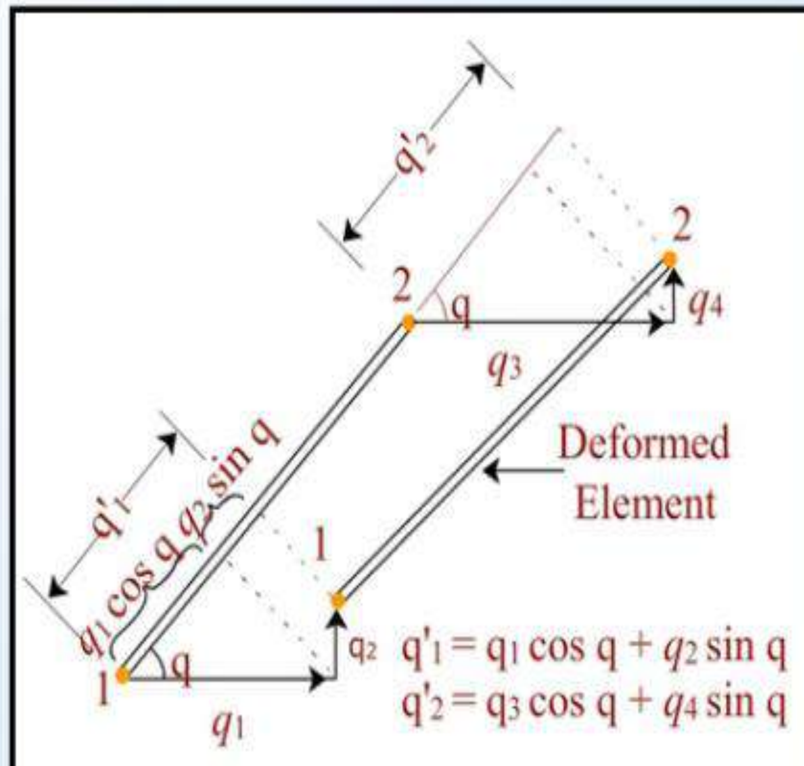
$$u = 0.00124 \; m$$

Finite Element Equations
(Assembly of Global Stiffness Matrix and Load Vector)

The main difference between the one-dimensional structures considered and trusses is that the elements of a truss have various orientations. To account for these different orientations, local and global coordinate systems are introduced.



Local Coordinate Systems



Global Coordinate Systems

Finite Element Equations
(Assembly of Global Stiffness Matrix and Load Vector)

1. In the local numbering scheme, the two nodes of the element are numbered 1 and 2.

2. The local coordinate system consists of the x' -axis, which runs along the element from node 1 toward node.

3. All quantities in the local coordinate system will be denoted by a prime (').

4. The global x-, y-coordinate system is fixed and does not depend on the orientation of the element.

5. Note that $x,y,$ and z form a right-handed coordinate system with the z -axis coming straight out of the paper.

6. In global Coordinate system every node has two degrees of freedom (dofs).

A systematic numbering scheme is adopted here: Anode whose global node number is j has associated with it dofs $2j - 1$ and $2j$: Further, the global displacements associated with node j are $Q\ 2j\text{-}1$ and $Q\ 2j$

Finite Element Equations
(Assembly of Global Stiffness Matrix and Load Vector)

- Let q_1 and q_2 be the displacements of nodes 1 and 2, respectively, in the local coordinate system. Thus, the element displacement vector in the. local coordinate system is denoted by

$$q' = [q_1', q_2']^T \qquad \text{--- 1}$$

- The element displacement vector in the global coordinate system is a (4×1) vector denoted by,

$$q = [q_1, q_2, q_3, q_4]^T \qquad \text{--- 2}$$

- The relationship between q' and q is developed as follows: that q_1 equals the sum of the projections of q_1 and q_2 onto the x' -axis. Thus,

$$\begin{aligned} q_1' &= q_1 \cos\theta + q_2 \sin\theta \\ q_2' &= q_3 \cos\theta + q_4 \sin\theta \end{aligned} \qquad \text{--- 3}$$

- Transformation matrix L is given by,

$$L = \begin{bmatrix} \ell & m & 0 & 0 \\ 0 & 0 & \ell & m \end{bmatrix} \qquad \text{--- (4)}$$

Finite Element Equations
(Assembly of Global Stiffness Matrix and Load Vector)

Calculation Of L and M

Simple formulas are now given for calculating the direction cosines e and m from nodal coordinate data. let (X_1, Y_1) and (X_2, Y_2) be the coordinates of nodes 1 and 2, respectively.

We then have,

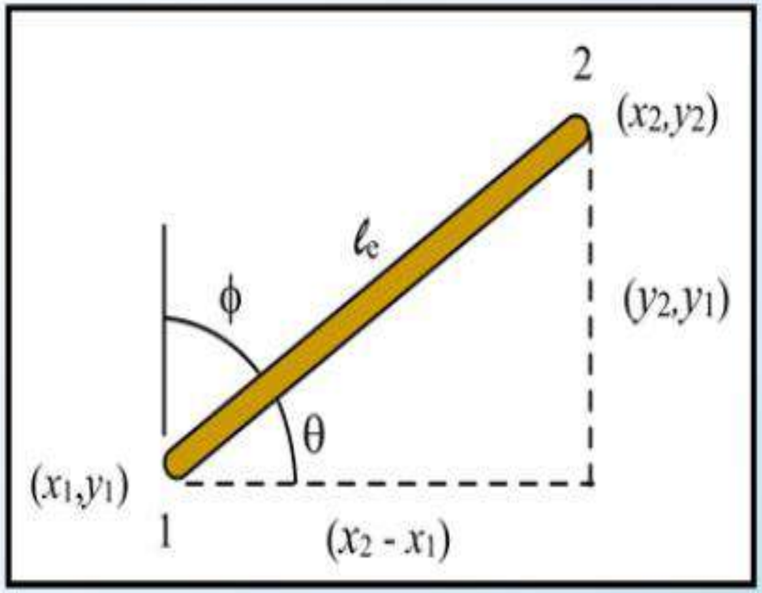
$$\ell = \cos\theta = \frac{x_2 - x_1}{\ell_e}$$

$$m = \cos\phi = \frac{y_2 - y_1}{\ell_e} (= \sin\theta)$$

$$\ell_e = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$\ell = \frac{x_2 - x_1}{\ell_e}$$

$$m = \frac{y_2 - y_1}{\ell_e}$$



The Equivalent length can be calculated by

$$\ell_e = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Finite Element Equations
(Assembly of Global Stiffness Matrix and Load Vector)

Element stiffness matrix

The truss element is a one-dimensional element when viewed in the local coordinate system.

$$k = \frac{E_e A_e}{\ell_e}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \qquad --- 5$$

where, A_e is the element cross-sectional area and E_e is Young's modulus. The problem at hand is to develop an expression for the element stiffness matrix in the global coordinate system. This is obtainable by considering the strain energy in the element. Specifically, the element strain energy in local coordinates is given by

$$U_e = \frac{1}{2}\mathbf{q}^{\mathrm{T}}\mathbf{k}\mathbf{q} \qquad --- 6$$

The stiffness matrix in global coordinates can be written as,

$$k = L^{T}K'L \qquad --- 7$$

Finite Element Equations
(Assembly of Global Stiffness Matrix and Load Vector)

The Stiffness matrix is given by,

$$\mathrm{k} = \frac{E_e A_e}{\ell_e}\begin{pmatrix} \ell^2 & \ell m & -\ell^2 & -\ell m \\ \ell m & m^2 & -\ell m & -m^2 \\ -\ell^2 & -\ell m & \ell^2 & \ell m \\ -\ell m & -m^2 & \ell m & m^2 \end{pmatrix} \qquad ---8$$

Stress Calculations:

Expressions for the element stress can be obtained by noting that a truss element in local coordinates is a simple two-force member

$$\sigma = E_e \frac{q_2' - q_1'}{\ell_e}$$

$$= \frac{E_e}{\ell_e}[-1 \quad 1]\begin{Bmatrix} q_1' \\ q_2' \end{Bmatrix} \qquad ---9$$

This equation can be written in terms of the global displacements q using the transformation $q' = Lq$ as,

$$\sigma = \frac{E_e}{\ell_e}[-1 \quad 1]\mathbf{Lq} \qquad ---10$$

Finite Element Equations
(Assembly of Global Stiffness Matrix and Load Vector)

The final Expression for the Stress calculation will be,

$$\sigma = \frac{E_e}{\ell}[-\ell \quad -m \quad \ell \quad \mathrm{m}]\mathbf{q} \qquad \text{- - - 11}$$

Problems on Truss

Consider the four-bar truss shown in Fig. E4.1a. It is given that E = 29.5 X 106 psi and A = 1 in 2 for all elements. Complete the following:

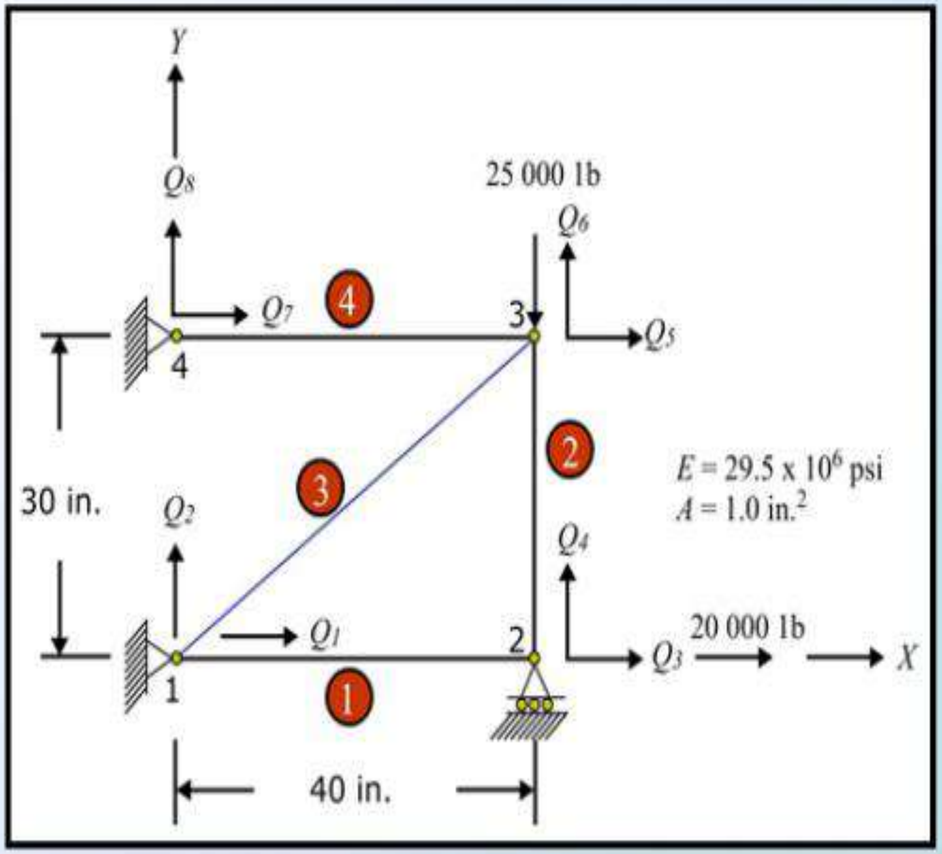
(a) Determine the element stiffness matrix for each element.

(b) Assemble the structural stiffness matrix K for the entire truss.

(c) Using the elimination approach, solve for the nodal displacement.

(d) Recover the stresses in each element.

(e) Calculate the reaction forces.

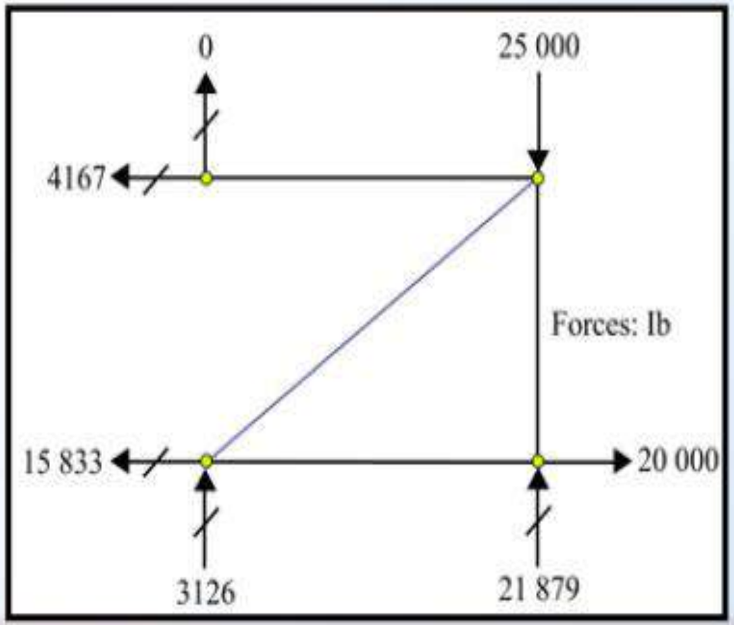


Problems on Truss

(a) It is recommended that a tabular form be used for representing nodal coordinate data and element information. The nodal coordinate data are as follows:

Node	*x*	*y*
1	0	0
2	40	0
3	40	30
4	40	30

For simplicity the diagram has been modified as follows,



Problems on Truss

The element connectivity table is,

Element	1	2
1	1	2
2	3	2
3	1	3
4	4	3

The nodal coordinate data and the given element connectivity information, we obtain the direction cosines table:

Element	ℓ_e	ℓ	m
1	40	1	0
2	30	0	-1
3	50	0.8	0.6
4	40	1	0

For example, the direction cosines of elements 3 are obtained as C = (X3 – X1)/le = (40 - 0)/50 = 0.8 and m = (Y3 – Y1)/le = (30 - 0)/50 = 0.6. The element stiffness matrices for element 1 can be written as

$$K^1 = \frac{29.5 \times 10^6}{40} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array} \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array}$$

Problems on Truss

The element stiffness matrices of elements 2,3, and 4 are as follows,

$$K^2 = \frac{29.5\times10^6}{30}\begin{array}{c}\begin{array}{cccc}5 & 6 & 3 & 4\end{array}\\ \begin{bmatrix}0 & 0 & 0 & 0\\ 0 & 1 & 0 & -1\\ 0 & 0 & 0 & 0\\ 0 & -1 & 0 & 1\end{bmatrix}\end{array}\begin{array}{c}\\5\\6\\3\\4\end{array}$$

$$K^3 = \frac{29.5\times10^6}{30}\begin{array}{c}\begin{array}{cccc}1 & 2 & 5 & 6\end{array}\\ \begin{bmatrix}.64 & .48 & -.64 & -.48\\ .48 & .36 & -.48 & -.36\\ -.64 & -.48 & .64 & .48\\ -.48 & -.36 & .48 & 36\end{bmatrix}\end{array}\begin{array}{c}\\1\\2\\5\\6\end{array}$$

$$K^4 = \frac{29.5\times10^6}{30}\begin{array}{c}\begin{array}{cccc}7 & 8 & 5 & 6\end{array}\\ \begin{bmatrix}1 & 0 & -1 & 0\\ 0 & 0 & 0 & 0\\ -1 & 0 & 1 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}\end{array}\begin{array}{c}\\7\\8\\5\\6\end{array}$$

Problems on Truss

(b) The structural stiffness matrix K is now assembled from the element stiffness matrices. By adding the element stiffness contributions, noting the element connectivity, we get,

$$K = \frac{29.5\times10^6}{600}\begin{array}{c} \begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{array} \\ \left[\begin{array}{cccccccc} 22.68 & 5.76 & -15.0 & 0 & -7.68 & -5.76 & 0 & 0 \\ 5.76 & 4.32 & 0 & 0 & -5.76 & -4.32 & 0 & 0 \\ -15.0 & 0 & 15.0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 20.0 & 0 & -20.0 & 0 & 0 \\ -7.68 & -5.76 & 0 & 0 & 22.68 & 5.76 & -15.0 & 0 \\ -5.76 & -4.32 & 0 & -20.0 & 5.76 & 24.32 & 0 & 0 \\ 0 & 0 & 0 & 0 & -15.0 & 0 & 15.0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \end{array} \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \end{array}$$

(c) The structural stiffness matrix K given above needs to be modified to account for the boundary conditions. The elimination approach will be used here. The rows and columns corresponding to dofs 1, 2, 4, 7 and 8, which correspond to fixed supports, are deleted from the K matrix.

Problems on Truss

The reduced finite element equations as follows,

$$\frac{29.5\times10^{6}}{600}\begin{bmatrix}15 & 0 & 0\\ 0 & 22.68 & 5.76\\ 0 & 5.76 & 24.32\end{bmatrix}\begin{Bmatrix}Q_3\\ Q_5\\ Q_6\end{Bmatrix}=\begin{Bmatrix}20\,000\\ 0\\ -25\,000\end{Bmatrix}$$

Solution of these equations yields the displacements,

$$\begin{Bmatrix}Q_3\\ Q_5\\ Q_6\end{Bmatrix}=\begin{Bmatrix}27.12\times10^{-3}\\ 5.65\times10^{-3}\\ -22.25\times10^{-3}\end{Bmatrix}\text{in.}$$

The nodal displacement vector for the entire structure can therefore be written as,

$$Q=[0,0,27.12\times10^{-3},0,5.65\times10^{-3},-22.25\times10^{-3},0,0]^{T}\text{ in.}$$

Problems on Truss

(d) The stress in each element can now be determined by,

$$\sigma_1 = \frac{29.5\times10^6}{40}[-1\ 0\ 1\ 0]\begin{Bmatrix} 0 \\ 0 \\ 27.12\times10^{-3} \\ 0 \end{Bmatrix}$$

$$= 20\ 000.0\text{psi}$$

The stress in member 2 is given by

$$\sigma_2 = \frac{29.5\times10^6}{30}[0\ 1\ 0\ -1]\begin{Bmatrix} 5.65\times10^{-3} \\ -22.25\times10^{-3} \\ -27.12\times10^{-3} \\ 0 \end{Bmatrix}$$

$$= -21\ 880.0\ \text{psi}$$

$$\sigma_3 = 5208.0\text{psi}$$

$$\sigma_4 = 4167.0\text{psi}$$

Problems on Truss

(e) The final step is to determine the support reactions. We need to determine the reaction forces along dofs 1,2,4,7, and 8, which correspond to fixed supports. These are obtained by substituting for Q into the original finite element equation R = KQ - F. In this substitution, only those rows of K corresponding to the support dofs are needed, and F = 0 for these dofs. Thus, we have

$$\begin{Bmatrix} R_1 \\ R_2 \\ R_3 \\ R_4 \\ R_5 \end{Bmatrix} = \frac{29.5\times10^6}{600}\begin{bmatrix} 22.68 & 5.76 & -15.0 & 0 & -7.68 & -5.76 & 0 & 0 \\ 5.76 & 4.32 & 0 & 0 & -5.76 & -4.32 & 0 & 0 \\ 0 & 0 & 0 & 20.0 & 0 & -20.0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -15.0 & 0 & 15.0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}\begin{Bmatrix} 0 \\ 0 \\ 27.12\times10^{-3} \\ 0 \\ 5.65\times10^{-3} \\ -22.25\times10^{-3} \\ 0 \\ 0 \end{Bmatrix}$$

$$\begin{Bmatrix} R_1 \\ R_2 \\ R_3 \\ R_4 \\ R_5 \end{Bmatrix} = \begin{Bmatrix} -15833.0 \\ 3126.0 \\ 21879.0 \\ -4167.0 \\ 0 \end{Bmatrix} lb$$

Treatment of Boundary Conditions in Truss

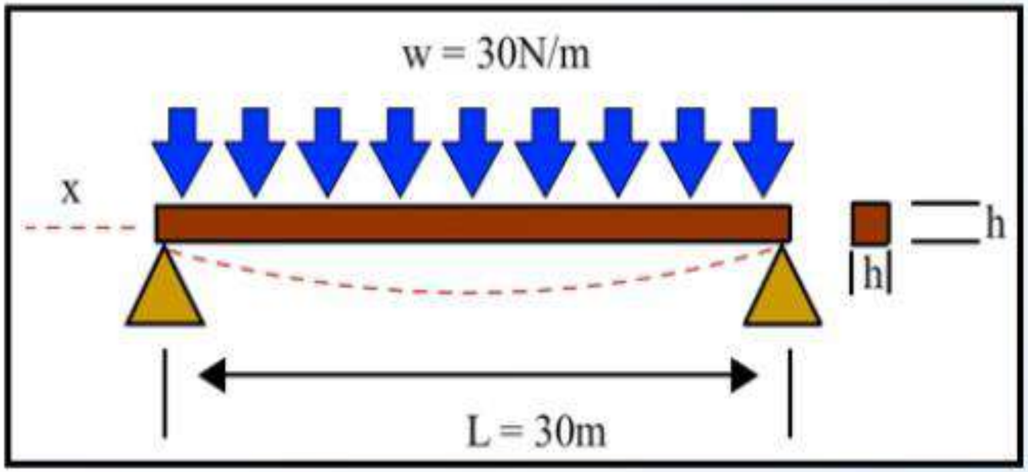
- The boundary conditions are the important aspects of the problem analysis. The boundary conditions has to be specified at every support particularly the truss is concern .

The boundary conditions has to be specified for,

- Simply supported condition
- Fixed condition
- Roller support condition
- Free end conditions

Treatment of Boundary Conditions in Truss

Simply Supported Condition



When the object is simply supported at both ends,

First end :

Displacement in x direction is finite

Displacement in y direction is zero

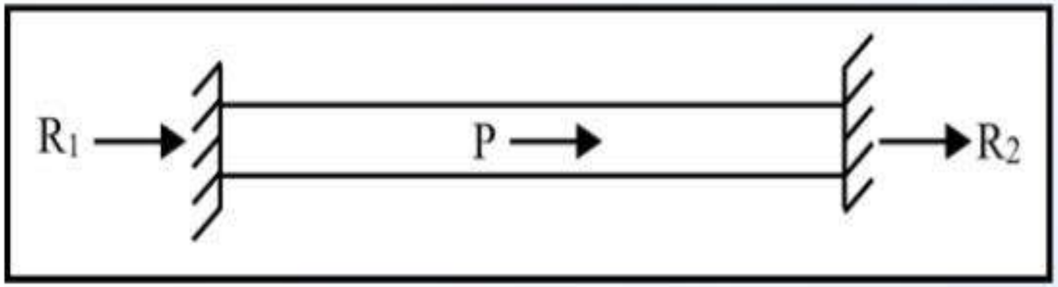
Second end :

Displacement in x direction is finite

Displacement in y direction is zero

Treatment of Boundary Conditions in Truss

Fixed Condition



When the object is fixed at both ends,

First end :

Displacement in x direction is zero

Displacement in y direction is zero

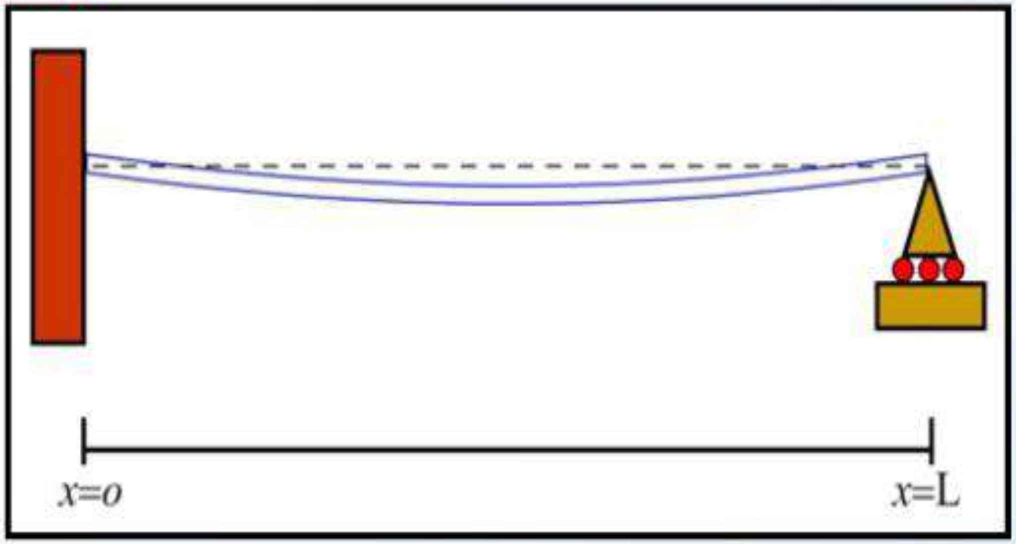
Second end :

Displacement in x direction is zero

Displacement in y direction is zero

Treatment of Boundary Conditions in Truss

Roller Support Condition



When the object is fixed at one end and roller supported at other end,

First end :

Displacement in x direction is zero

Displacement in y direction is zero

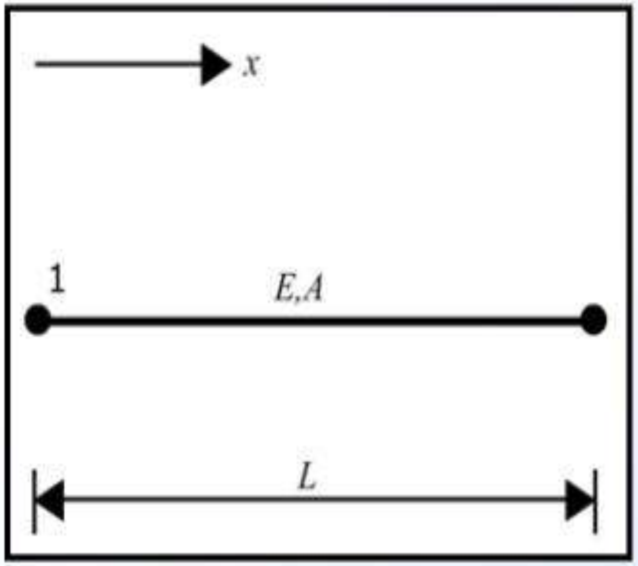
Second end :

Displacement in x direction is finite

Displacement in y direction is zero

Treatment of Boundary Conditions in Truss

Free End Conditions



When the object is free at both ends,

First end :

Displacement in x direction is finite

Displacement in y direction is finite

Second end :

Displacement in x direction is finite

Displacement in y direction is finite

Summary

Let's summarize the topic:

- A truss structure consists only of two-force members. That is, every truss element is in direct tension or compression .
- The warren truss has made up of several bar pin jointed or welded. At the final stage the ends will be either roller supported or pinned supported. It has a good strength.
- Discritization is a process of subdividing the whole structure into a small finite elements of equal size and shape for the better accurate solutions of the object
- The main difference between the one-dimensional structures considered and trusses is that the elements of a truss have various orientations.

Reference

Books:

1. An introduction to Finite Element Method / JN Reddy / McGrawHill

2. The Finite Element Method for Engineers – Kenneth H. Huebner, Donald L. Dewhirst, Douglas E. Smith and Ted G. Byrom / John Wiley & sons (ASIA) Pte Ltd.

3. Finite Element Analysis: Theory and Application with Ansys, Saeed Moaveniu, Pearson Education

Analysis of Beams

Learning Objectives

At the end of this topic, you will be able to:

- Describe the Potential energy approach
- Describe the Galerkin approach
- Illustrate the Element Stiffness Matrix for Hermite Beam Element
- Derive load vector derivation for concentrated and UDL

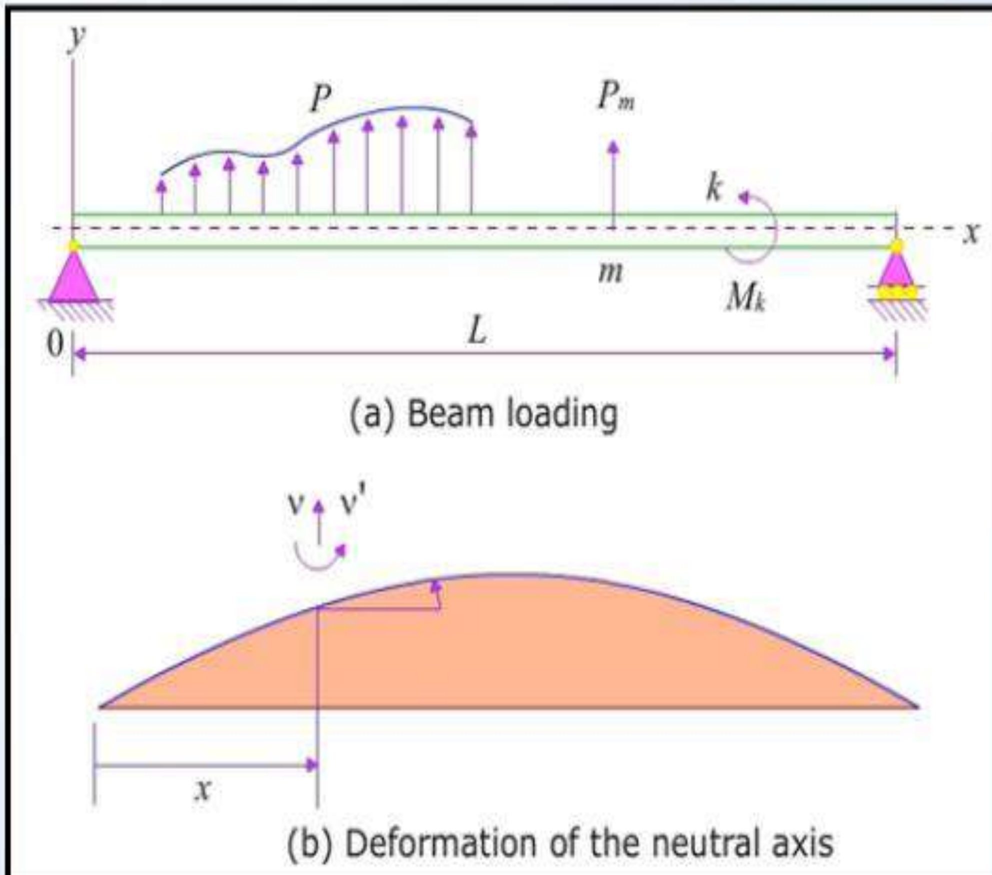
Outcomes

By the end of this topic, you will be able to:

- Understand the Potential energy approach
- Discuss about the Galerkin approach
- Explain the Element Stiffness Matrix for Hermite Beam Element
- Workout the load vector derivation for concentrated and UDL

Introduction

- Beams are slender members that are used for supporting transverse loading.
- Long horizontal members used in buildings and bridges, and shafts supported in bearings are some examples of beams.
- Beams with cross sections that are symmetric with respect to plane of loading are considered here.
- A general horizontal beam is shown.
- Figure shows the cross section and the bending stress distribution.

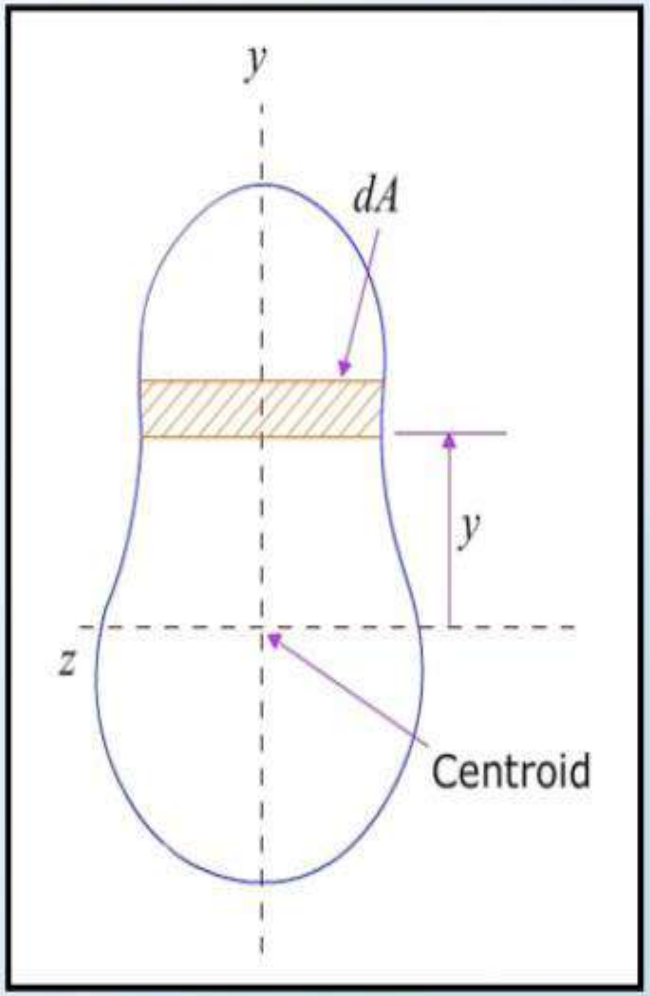


(a) Beam loading

(b) Deformation of the neutral axis

Introduction

- For small deflections, we recall from elementary beam theory that where is the normal stress, is the normal strain, M is the bending moment at the section, is the deflection of the centroidal axis at x, and I is the moment of inertia of the section about the neutral axis (z-axis passing through the centroid).



$$\sigma = -\frac{M}{I} y \qquad \text{---(1)}$$

$$\in = \frac{\sigma}{E} \qquad \text{---(2)}$$

$$\frac{d^2 v}{dx^2} = \frac{M}{EI} \qquad \text{---(3)}$$

Potential - Energy Approach

The strain energy in an element of length dx is

$$dU = \frac{1}{2}\int_A \sigma \in dA\, dx$$

$$= \frac{1}{2}\left(\frac{M^2}{EI^2}\int_A y^2 dA\right) dx$$

Noting that ${}_A y^2\, dA$ is the moment of inertia I, we have

$$dU = \frac{1}{2}\frac{M}{EI}dx \qquad (4)$$

When Eq. 3 is used, the total strain energy in the beam is given by

$$U = \frac{1}{2}\int_0^L EI\left(\frac{d^2v}{dx^2}\right)^2 dx \qquad (5)$$

Potential - Energy Approach

The potential energy of the beam is then given by

$$\Pi = \frac{1}{2}\int_0^L EI\left(\frac{d^2v}{dx^2}\right)^2 dx - \int_0^L pv\,dx - \sum_m P_m v_m - \sum_k M_k v'_k \qquad (6)$$

where p is the distributed load per unit length, P_m is the point load at point m, M_k is the moment of the couple applied at point k, v_m is the deflection at point m, and v'_k is the slope at point k.

Galerkin Approach

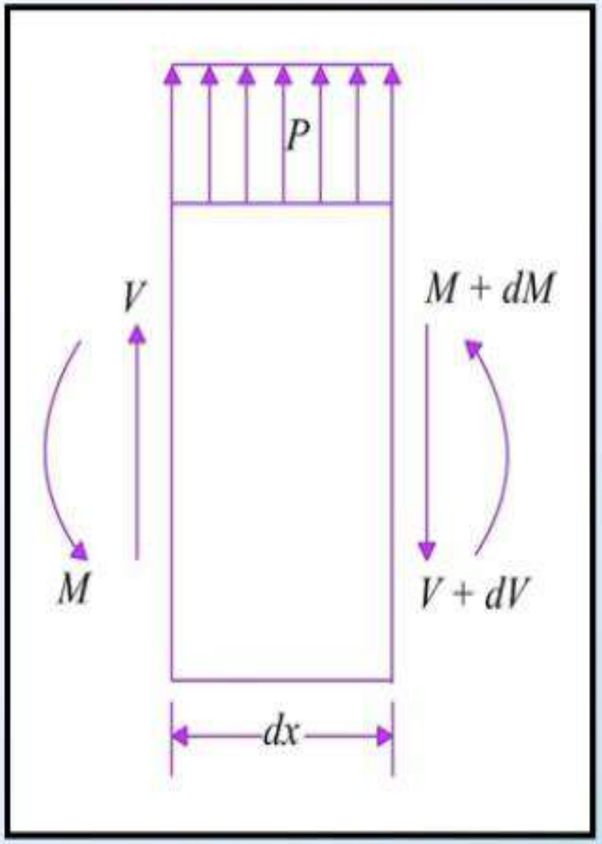
For the Galerkin formulation, we start from equilibrium of an elemental length. From we recall that

$$\frac{dV}{dx} = p \qquad (7)$$

$$\frac{dM}{dx} = V \qquad (8)$$

When Eqs. 3, 7, and 8 are combined, the equilibrium equation is given by

$$\frac{d^2}{dx^2}\left(EI\frac{d^2v}{dx^2}\right) - p = 0 \qquad (9)$$



Free body diagram of an elemental length *dx*.

For approximate solution by the Galerkin approach, we look for the approximate solution v constructed of finite element shape functions such that

Galerkin Approach

$$\int_0^L \left[\frac{d}{dx^2}\left(EI \frac{d^2 v}{dx^2} \right) - p \right] \phi \, dx = 0 \qquad (10)$$

Where ϕ is an arbitrary function using same basis functions as v. Note that ϕ is zero where has v a specified value. We integrate the first term of Eq. 10 by parts. The integral from 0 to L is split into intervals 0 to X_m, X_m to X_b and X_k to L. We obtain

$$\int_0^L EI \frac{d^2 v}{dx^2}\frac{d^2 \Phi}{dx^2} dx - \int_0^L p\phi dx + \frac{d}{dx}\left(EI \frac{d^2 v}{dx^2} \right)\phi \Big|_0^{x_m} + \frac{d}{dx}\left(EI \frac{d^2 v}{dx^2} \right)\phi \Big|_{x_m}^{L}$$

$$- EI \frac{d^2 v}{dx^2}\frac{d\phi}{dx}\Big|_0^{x_k} - EI \frac{d^2 v}{dx^2}\frac{d\phi}{dx}\Big|_{x_k}^{L} = 0 \qquad (11)$$

We note that $EI(d^2 v / dx^2)$ equals the bending moment M from Eq. 3 and $(d/dx)[EI(d^2 v/dx^2)]$ equals the shear force ϕ from (8). Also, v and M are zero at the supports.

Galerkin Approach

At X_m , the jump in shear force is P_m and at X_b the jump in bending moment is - M_k. Thus, we get

$$\int_0^L EI\frac{d^2v}{dx^2}\frac{d^2\phi}{dx^2}dx - \int_0^L p\phi dx - \sum_m P_m\phi_m - \sum_k M_k\phi_k' = 0 \qquad (12)$$

For the finite element formulation based on Galerkin's approach, v and ϕ are constructed using the same shape functions. Equation 12 is precisely the statement of the principle of virtual work.

Element Stiffness Matrix for Hermite Beam Element

Finite Element Formulation

- Typically, the degrees of freedom of node *i* are Q_{2i-1} and Q_{2i}. The degree of freedom Q_{2i-1} is transverse displacement and Q_{2i} is slope or rotation. The vector represents the global displacement vector.

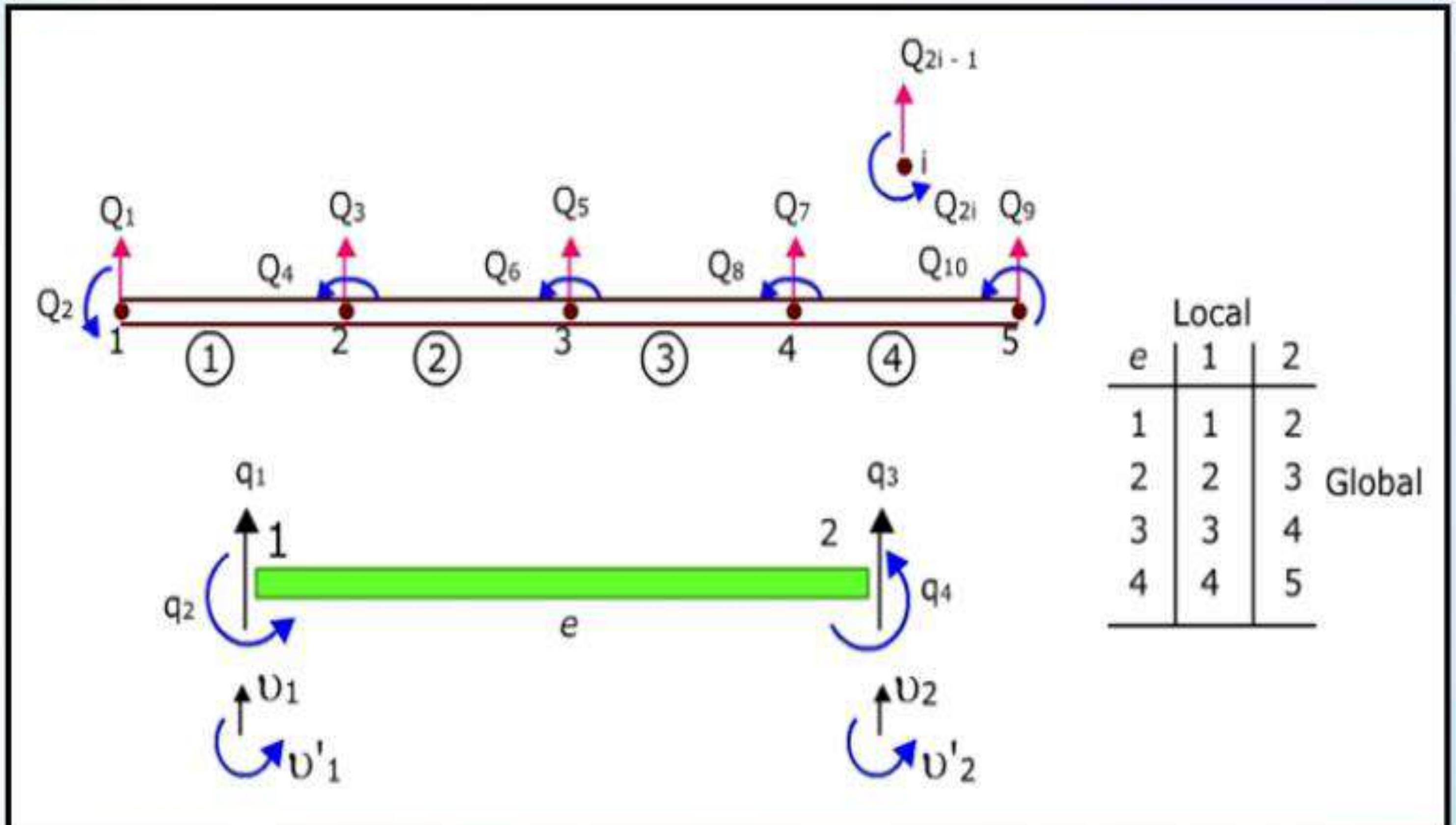
$$Q = [Q_1, Q_2, \ldots Q_{10}]^T \qquad \text{---(13)}$$

- For a single element, the local degrees of freedom are represented by

$$q = [q_1, q_2, q_3, q_4]^T \qquad \text{---(14)}$$

Element Stiffness Matrix for Hermite Beam Element

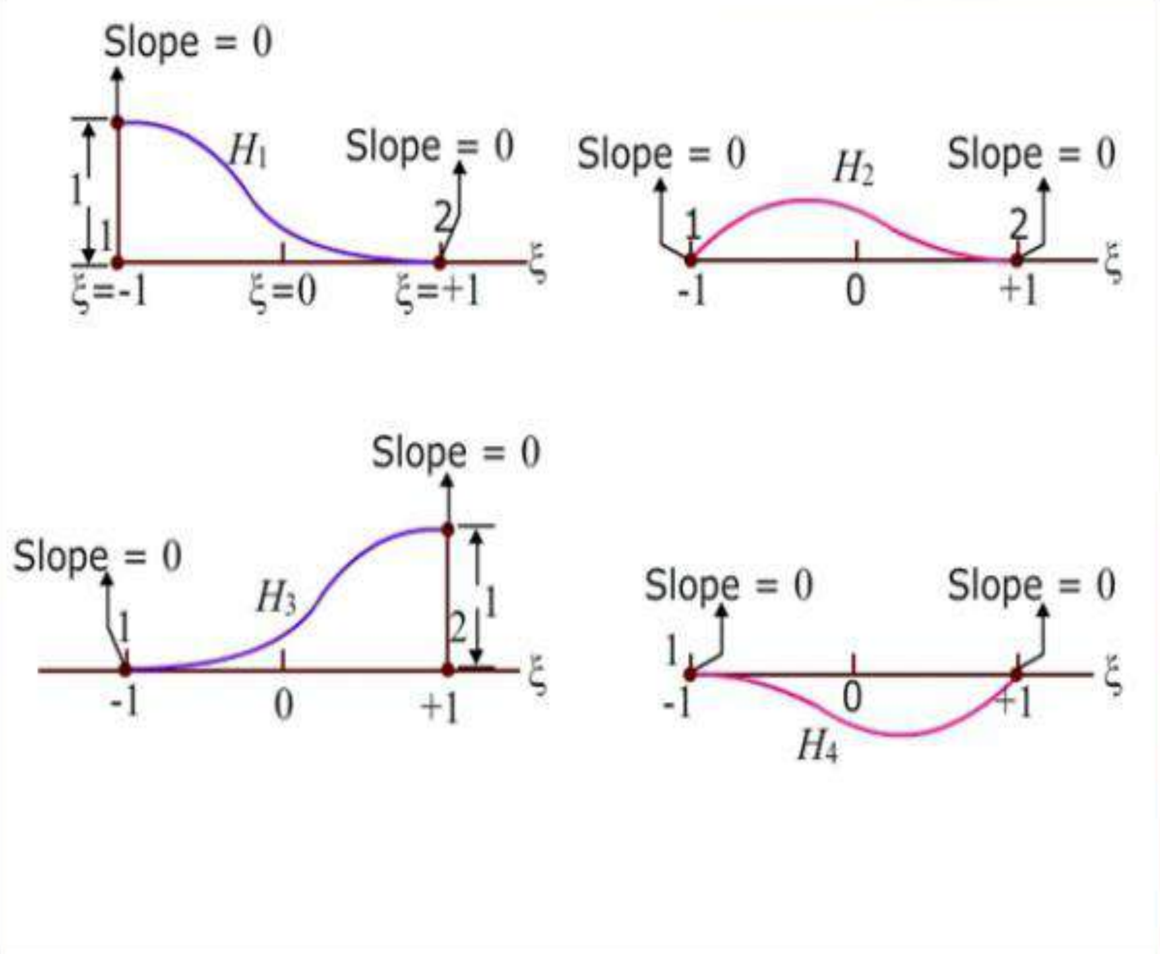
- The beam is divided into elements, as shown in Fig. Each node has two degrees of freedom.



- The local-global correspondence is easy to see from the table given in Fig.(Finite element discretization) q is same as $[v_1, v'_1, v_2, v'_2]^T$

Element Stiffness Matrix for Hermite Beam Element

- The shape functions for interpolating v on an element are defined in terms of ξ on -1 to + 1, as shown in Fig.(Hermite shape functions).



Hermite shape functions

Element Stiffness Matrix for Hermite Beam Element

- The shape functions for beam elements differ from those discussed earlier. Since nodal values and nodal slopes are involved, we define Hermite shape functions, which satisfy nodal value and slope continuity requirements. Each of the shape functions is of cubic order represented by

$$H_i = a_i + b_i\xi + c_i\xi^2 + d_i\xi^3, \qquad i = 1,2,3,4 \qquad \text{---(15)}$$

- The conditions given in the following table must be satisfied:

	H_1	H'_1	H_2	H'_2	H_3	H'_3	H_4	H'_4
$\xi = -1$	1	0	0	1	0	0	0	0
$\xi = 1$	0	0	0	0	1	0	0	1

Element Stiffness Matrix for Hermite Beam Element

- The coefficients a_j, b_j, c_j, and d_j can be easily obtained by imposing these conditions.

  Thus,

$$
\begin{aligned}
H_1 &= \frac{1}{4}(1-\xi)^2(2+\xi) \text{ or } \frac{1}{4}\left(2-3\xi+\xi^3\right)\\
H_2 &= \frac{1}{4}(1-\xi)^2(\xi+1) \text{ or } \frac{1}{4}\left(1-\xi-\xi^2+\xi^3\right)\\
H_3 &= \frac{1}{4}(1+\xi)^2(2-\xi) \text{ or } \frac{1}{4}\left(2+3\xi-\xi^3\right)\\
H_4 &= \frac{1}{4}(1+\xi)^2(\xi-1) \text{ or } \frac{1}{4}\left(-1-\xi+\xi^2+\xi^3\right)
\end{aligned}
\qquad \text{---(16)}
$$

- The Hermite shape functions can be used to write v in the form

$$
v(\xi) = \mathrm{H}_1 v_1 + H_2\left(\frac{dv}{d\xi}\right)_1 + H_3 v_2 + H_4\left(\frac{dv}{d\xi}\right)_2 \qquad \text{---(17)}
$$

Element Stiffness Matrix for Hermite Beam Element

- The coordinates transform by the relationship

$$x = \frac{1-\xi}{2}x_1 + \frac{1+\xi}{2}x_2$$
$$= \frac{x_1 + x_2}{2} + \frac{x_2 - x_1}{2}\xi \qquad \text{---(18)}$$

Since $\ell_e = X_2 - X_1$ is the length of the element, we have

$$dx = \frac{\ell_e}{2}d\xi \qquad \text{---(19)}$$

- The chain rule $dv/d\xi = (\mathrm{dv/dx})(\mathrm{dx/d}\xi)$ gives us

$$\frac{dv}{d\xi} = \frac{\ell_e}{2}\frac{dv}{dx} \qquad \text{---(20)}$$

Element Stiffness Matrix for Hermite Beam Element

- Noting that dv / dx evaluated at nodes 1 and 2 is q_2 and q_4, respectively, we have

$$v(\xi) = \mathrm{H}_1 q_1 + \frac{\ell_e}{2} H_2 q_2 + H_3 q_3 + \frac{\ell_e}{2} H_4 q_4 \qquad \text{---(21)}$$

which may be denoted as

$$v = H q \qquad \text{---(22)}$$

where

$$H = \left[H_1, \frac{\ell_e}{2} H_2, H_3, \frac{\ell_e}{2} H_4 \right] \qquad \text{---(23)}$$

- In the total potential energy of the system, we consider the integrals as summations over the integrals over the elements. The element strain energy is given by

$$U_e = \frac{1}{2} EI \int_e \left(\frac{d^2 v}{dx^2} \right)^2 dx \qquad \text{---(24)}$$

Element Stiffness Matrix for Hermite Beam Element

From Eq. 20,

$$\frac{dv}{dx} = \frac{2}{\ell_e}\frac{dv}{d\xi} \quad and \quad \frac{d^2v}{dx^2} = \frac{4}{\ell_e^2}\frac{d^2v}{d\xi^2}$$

Then, substituting $v = Hq$, we obtain

$$\left(\frac{d^2v}{dx}\right)^2 = q^T\frac{16}{\ell_e^4}\left(\frac{d^2H}{d\xi^2}\right)^T\left(\frac{d^2H}{d\xi^2}\right)q \qquad \text{---(25)}$$

$$\left(\frac{d^2H}{d\xi^2}\right) = \left[\frac{3}{2}\xi, \frac{-1+3\xi}{2}, -\frac{3}{2}\xi, \frac{1+3\xi}{2}\frac{\ell_e}{2}\right] \qquad \text{---(26)}$$

Element Stiffness Matrix for Hermite Beam Element

On substituting $dx = (\ell_e / 2)d\xi$ and Eqs.25 and 26 in Eq.24, we get

$$U_e = \frac{1}{2}q^T \frac{8EI}{\ell_e^3}\int_{-1}^{+1}\begin{bmatrix} \frac{9}{4}\xi^2 & \frac{3}{8}\xi(-1+3\xi)\ell_e & -\frac{9}{4}\xi^2 & \frac{3}{8}\xi(1+3\xi)\ell_e \\ & \left(\frac{-1+3\xi}{4}\right)^2 \ell_e^2 & -\frac{3}{8}\xi(-1+3\xi)\ell_e & \frac{-1+9\xi^2}{16}\ell_e^2 \\ symmetric & & \frac{9}{4}\xi^2 & -\frac{3}{8}\xi(1+3\xi)\ell_e \left(\frac{1+3\xi}{4}\right)^2 \ell_e^2 \end{bmatrix} d\xi q \quad \text{---(27)}$$

Each term in the matrix needs to be integrated. Note that

$$\int_{-1}^{+1} \xi^2 d\xi = \frac{2}{3} \quad \int_{-1}^{+1} \xi\, d\xi = 0 \quad \int_{-1}^{+1} d\xi = 2$$

Element Stiffness Matrix for Hermite Beam Element

- This results in the element strain energy given by

$$U_e = \frac{1}{2} q^T k^e q \qquad \text{---(28)}$$

where the element stiffness matrix is

$$k^e = \frac{EI}{\ell_e^3} \begin{bmatrix} 12 & 6\ell_e & -12 & 6\ell_e \\ 6\ell_e & 4\ell_e^2 & -6\ell_e & 2\ell_e^2 \\ -12 & -6\ell_e & 12 & -6\ell_e \\ 6\ell_e & 2\ell_e^2 & -6\ell_e & 4\ell_e^2 \end{bmatrix} \qquad \text{---(29)}$$

which is symmetric.

Element Stiffness Matrix for Hermite Beam Element

- In the development based on Galerkin's approach (see Eq.12),

$$EI\frac{d^2\phi}{dx^2}\frac{d^2v}{dx^2}=\psi^T EI\frac{16}{\ell_e^4}\left(\frac{d^2H}{d\xi^2}\right)^T\left(\frac{d^2H}{d\xi^2}\right)q \qquad \text{---(30)}$$

Where

$$\psi=[\psi_1\ \psi_2\ \psi_3\ \psi_4]^T \qquad \text{---(31)}$$

- We note that is the set of generalized virtual displacements on the element, $v = Hq$ and $\phi = H\psi$ Equation 30 yields the same element stiffness as Eq. 28 on integration, with $\psi^T k^e q$ being the internal virtual work in an element.

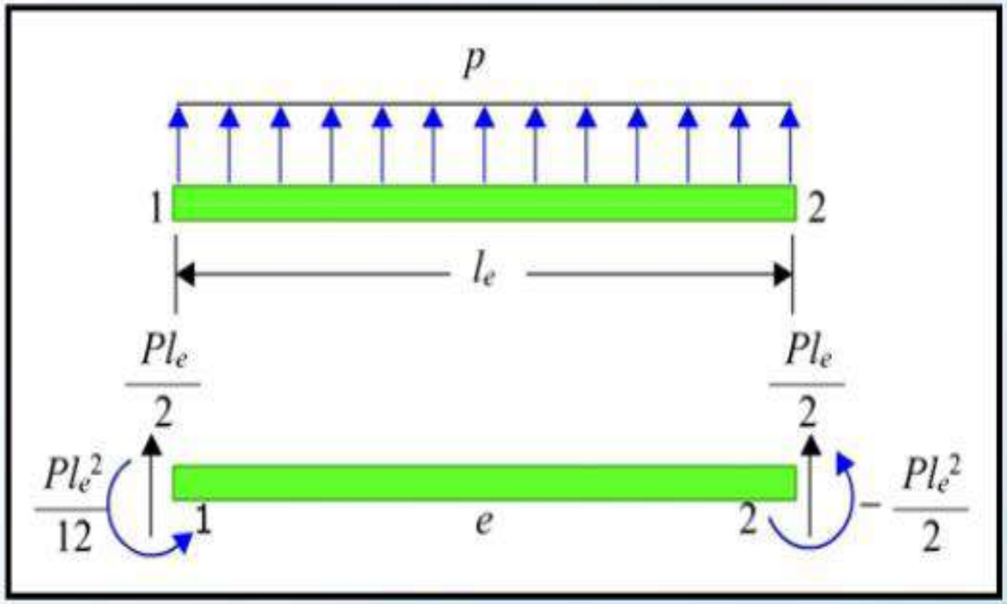
Derivation of Load Vector for Concentrated and UDL

- The load contributions from the distributed load p in the element is first considered. We assume that the distributed load is uniform over the element:

$$\int_{\ell_e} pv\,dx = \left(\frac{p\ell_e}{2}\int_{-1}^{+1} H\,d\xi\right) q \qquad \text{---(32)}$$

- On substituting for H from Eqs.16 and 23 and integrating, we obtain

$$\int_{\ell_e} pv\,dx = f^{eT} q \qquad \text{---(33)}$$



Derivation of Load Vector for Concentrated and UDL

Where

$$f^e = \left[\frac{p\ell_e}{2}, \frac{p\ell_e^2}{12}, \frac{p\ell_e}{2}, -\frac{p\ell_e^2}{12} \right]^T \qquad \text{---(34)}$$

❖ This equivalent load on an element is shown in Fig. 8.6. The same result is obtained by considering the term 1. $pc/> dx$ in Eq. 8.12 for the Galerkin formulation. The point loads P_m and M_k are readily taken care of by introducing nodes at the points of application. On introducing the local-global correspondence, from the potential-energy approach, we get

$$\Pi = \frac{1}{2} Q^T K Q - Q^T F \qquad \text{---(35)}$$

and from Galerkin's approach, we get

$$\Psi^T K Q - \Psi^T F = 0 \qquad \text{---(36)}$$

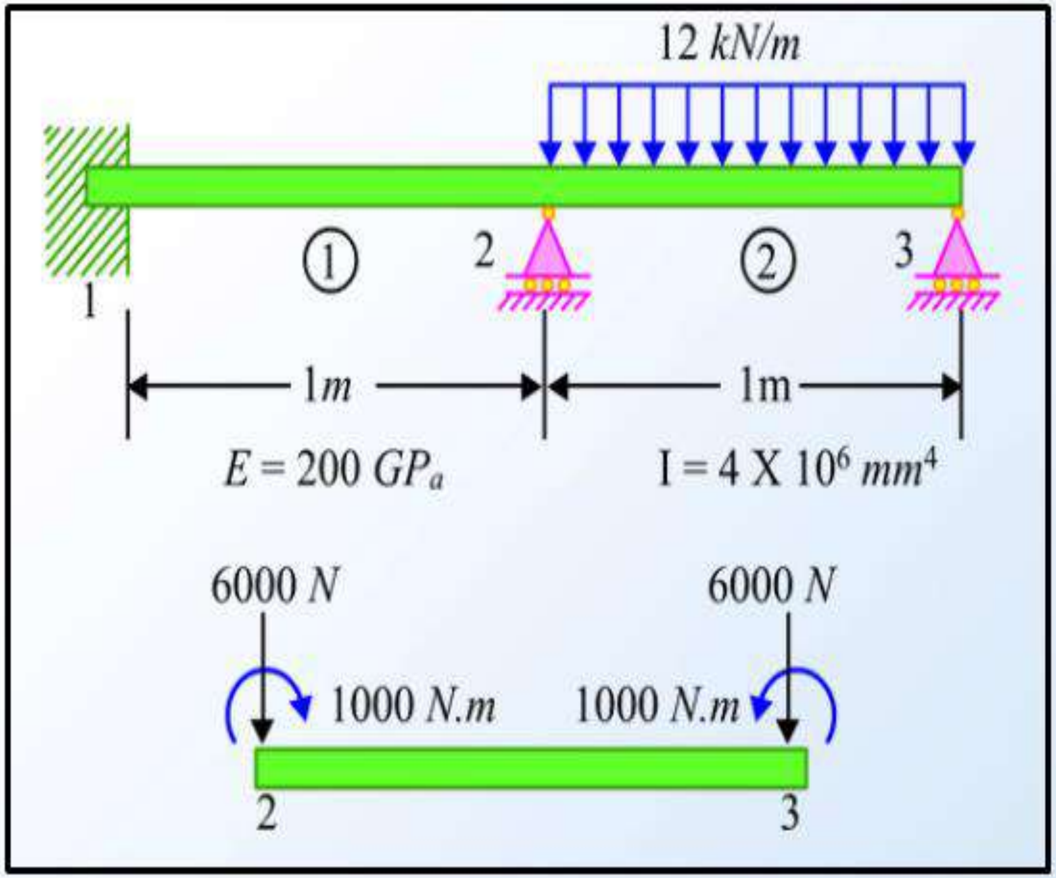
where Ψ = arbitrary admissible global virtual displacement vector.

Example-1 (Simple Problems on Beams)

For the beam and loading shown in fig, determine

1) The slopes at 2 and 3,

2) The vertical deflection at the midpoint of the distributed load.



Example-1

We consider the two elements formed by the three nodes. Displacements Q_1 ,Q_2, Q_3, and Q_5 are constrained to be zero, and Q_4 and Q_6 need to be found. Since the lengths and sections are equal, the element matrices are calculated from eq. 29 as follows:

$$\frac{EI}{\ell^3} = \frac{(200\times10^9)(4\times10^{-6})}{1^3} = 8\times10^5\ N/m$$

$$k^{1} = k^{2} = 8\times10^5 \begin{bmatrix} 12 & 6 & -12 & 6 \\ 6 & 4 & -6 & 2 \\ -12 & -6 & 12 & -6 \\ 6 & 2 & -6 & 4 \end{bmatrix}$$

$$\begin{array}{lcccc} e=1 & Q_1 & Q_2 & Q_3 & Q_4 \\ e=2 & Q_3 & Q_4 & Q_5 & Q_6 \end{array}$$

Example-1

We note that global applied loads are $F_4 = -1000\ N.m$ and $F_6 = +\ 1000\ N.m$ obtained from $pl^2/12$ as seen in fig. We use here the elimination approach. Using the connectivity, we obtain the global stiffness after elimination: $p\ell^2/12$

$$k = \begin{bmatrix} k_{44}^{(1)} + k_{22}^{(2)} & k_{24}^{(2)} \\ k_{42}^{(2)} & k_{44}^{(2)} \end{bmatrix}$$

$$= 8 \times 10^5 \begin{bmatrix} 8 & 2 \\ 2 & 4 \end{bmatrix}$$

The set of equations is given by

$$8 \times 10^5 \begin{bmatrix} 8 & 2 \\ 2 & 4 \end{bmatrix} \begin{Bmatrix} Q_4 \\ Q_6 \end{Bmatrix} = \begin{Bmatrix} -1000 \\ +1000 \end{Bmatrix}$$

Example-1

The solution is $\begin{Bmatrix} Q_4 \\ Q_6 \end{Bmatrix} = \begin{Bmatrix} -2.679\times10^{-4} \\ 4.464\times10^{-4} \end{Bmatrix}$

For element 2, $q_1 = 0$, $q_2 = Q_4$, $q_3 = 0$, and $q_4 = Q_6$' To get vertical deflection at the midpoint of the element, use $v = Hq \ at \ \xi = 0$

$$v = 0 + \frac{\ell_e}{2} H_2 Q_4 + 0 + \frac{\ell_e}{2} H_4 Q_6$$

$$= \left(\frac{1}{2}\right)\left(\frac{1}{4}\right)\left(-2.679\times10^{-4}\right) + \left(\frac{1}{2}\right)\left(-\frac{1}{4}\right)\left(4.464\times10^{-4}\right)$$

$$= -8.93\times10^{-5} \ m$$

$$= -0.0893 \ mm$$

Example-2

A beam element is subjected to a uniformly distributed transverse load along its length as shown in figure (a). Determine the consistent load vector using the basic concepts of solid mechanics.

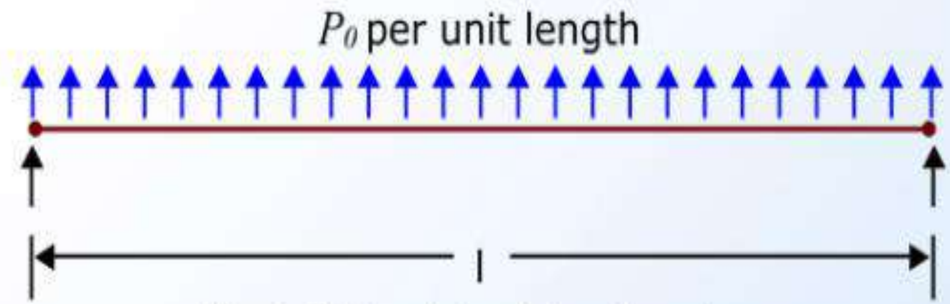


Fig (a) Consistent load vector

When a uniform beam is subjected to a uniformly distributed load of intensity P_o along its length l, the reactions at the ends given by the simple beam theory, when both the ends are fixed, is shown in figure (a).

Example-2

The negative values of the reactions can be taken as the nodal loads corresponding to a uniform beam element as shown in figure (b).

Fig (b) Consistent load vector

So that the element load vector due to a uniform distributed load can be taken as

$$\vec{p}^{(e)} = \begin{Bmatrix} \frac{p_0 l}{2} \\ \frac{p_0 l^2}{12} \\ \frac{p_0 l}{2} \\ -\frac{p_0 l^2}{12} \end{Bmatrix} \longrightarrow \text{(E.1)}$$

Example-2

The element load vectors of a fixed-fixed beam for other types of loads such as a linearly varying distributed load or a concentrated load or moment acting at any point along the length of the beam element can be generated in a similar manner using results from the simple beam theory.

The consistent load vector of a beam element corresponding to a specified load distribution can be generated.

Example-3

A uniformly distributed load of magnitude P_o per unit length is applied along the length of a beam element. Derive the corresponding consistent load vector in the local coordinate system.

Solution

The consistent load vector, $\overrightarrow{P_c}^{(e)}$ for a uniformly distributed load,. $\Phi(x) = P_0$

$$\overrightarrow{p_s}^{(e)} = \iint_{s_1^{(e)}} [N]^T dx = \int_{x=0}^{l^{(e)}} p_0 [N]^T dx = \int_{\overrightarrow{p_s}^{(e)}}^{l^{(e)}} p_0 \begin{Bmatrix} N_1(x) \\ N_2(x) \\ N_3(x) \\ N_4(x) \end{Bmatrix} dx$$

$$= \int_{x=0}^{l^{(e)}} p_0 \begin{Bmatrix} (2x^2 - 3lx^2 + l^3)/l^3 \\ (x^3 - 2lx^2 + l^2 x)/l^2 \\ -(2x^2 - 3lx^2)/l^3 \\ (x^3 - lx^2)/l^2 \end{Bmatrix} dx = \begin{Bmatrix} \dfrac{p_0 l}{2} \\ \dfrac{p_0 l^2}{12} \\ \dfrac{p_0 l}{2} \\ -\dfrac{p_0 l^2}{12} \end{Bmatrix} \longrightarrow \quad \text{(E.1)}$$

Which can be seen to be identical to eq. (e. 1) of example 2.

Summary

Let's summarize the topic:

- Beams are slender members that are used for supporting transverse loading.
- Long horizontal members used in buildings and bridges, and shafts supported in bearings are some examples of beams.
- Analysis of beams is carried out by Potential energy approach and Galerkin approach.

Reference

Books:

1. An introduction to Finite Element Method / JN Reddy / McGrawHill

2. The Finite Element Method for Engineers – Kenneth H. Huebner, Donald L. Dewhirst, Douglas E. Smith and Ted G. Byrom / John Wiley & sons (ASIA) Pte Ltd.

3. Finite Element Analysis: Theory and Application with Ansys, Saeed Moaveniu, Pearson Education

Two Dimensional Stress Analysis - I

Learning Objectives

At the end of this topic, you will be able to:

- Understand the two dimensional finite element formulation
- Explain about finite element modelling
- Describe constant strain triangle
- Illustrate the Iso-parametric representation
- Solve the Jacobian of transformation
- Explain the potential energy approach
- Explain Element stiffness and Force terms
- Discuss the Characteristics of constant strain triangle element
- Understand the Treatment of boundary conditions

Outcomes

By the end of this topic, you will be able to:

- Explain two dimensional finite element formulation
- Understand about finite element modeling and constant strain triangle
- Discuss the Iso-parametric representation
- Workout the Jacobian of transformation
- Discuss the potential energy approach
- Describe about Element stiffness and Force terms
- Understand the Characteristics of constant strain triangle element
- Explain the Treatment of boundary conditions

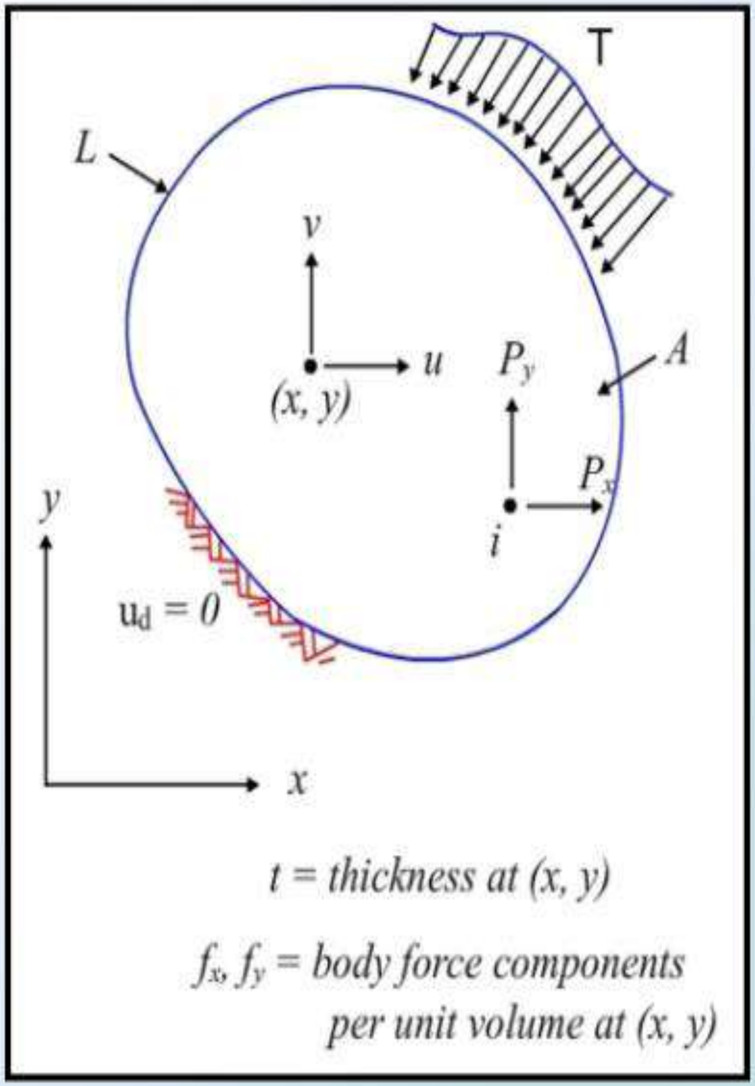
Two Dimensional Finite Element Formulation

- The displacements, traction components, and distributed body force values are functions of the position indicated by (x, y). The displacement vector u_d is given as,

$$u_d = [u, v]^T$$

Where, *u* and *v* are the *x* and *y* components of u_d, respectively.

- The two dimensional problem in a general setting for an arbitrary shape is shown in figure



Two Dimensional Finite Element Formulation

- The body force, traction vector, and elemental volume are given by,

$$f = [f_x, fy]T \qquad T=[T_x, T_y]T \quad \text{and} \quad dV = t\ dA$$

Where, t is the thickness along the z direction.

- The body force f has the units force/unit volume, while the traction force T has the units *force/unit area*. The strain-displacement relations are given by,

$$\in = \left[\frac{du}{dx}, \frac{dv}{dy}, \left(\frac{du}{dy} + \frac{dv}{dx} \right) \right]^T$$

- Stresses and strains are related by,

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \frac{E}{1-v^2} \begin{bmatrix} 1 & v & 0 \\ v & 1 & 0 \\ 0 & 0 & \frac{1-v}{2} \end{bmatrix} \begin{Bmatrix} \in_x \\ \in_y \\ \gamma_{xy} \end{Bmatrix}$$

Two Dimensional Finite Element Formulation

Where,

- σ_x, σ_y - represent stresses in x and y directions (N/m^2).
- τ_{xy} Represent shear stress in x-y direction (N/m^2).
- E – modulus of elasticity (N/m^2).
- v - Poisson's ratio.
- $\in_x, \in_y$ - strain in x and y directions (N/m^2) ,
- γ_{xy} - shear strain in x-y direction (N/m^2).

Two Dimensional Finite Element Formulation

- In general the above equation can be represented as,

$$\sigma = D\epsilon$$

where D here is a $(3 \; X \; 3)$ matrix, which relates three stresses and three strains.

- The region is discretized with the idea of expressing the displacements in terms of values at discrete points.
- Triangular elements are introduced first.
- Stiffness and load concepts are then developed using energy and Galerkin approaches.

Finite Element Modeling

- The two-dimensional region is divided into straight-sided triangles. Figure.1 shows a typical triangulation.
- The points where the corners of the triangles meet are called nodes, and each triangle formed by three nodes and three sides is called an element.
- The elements fill the entire region except a small region at the boundary.

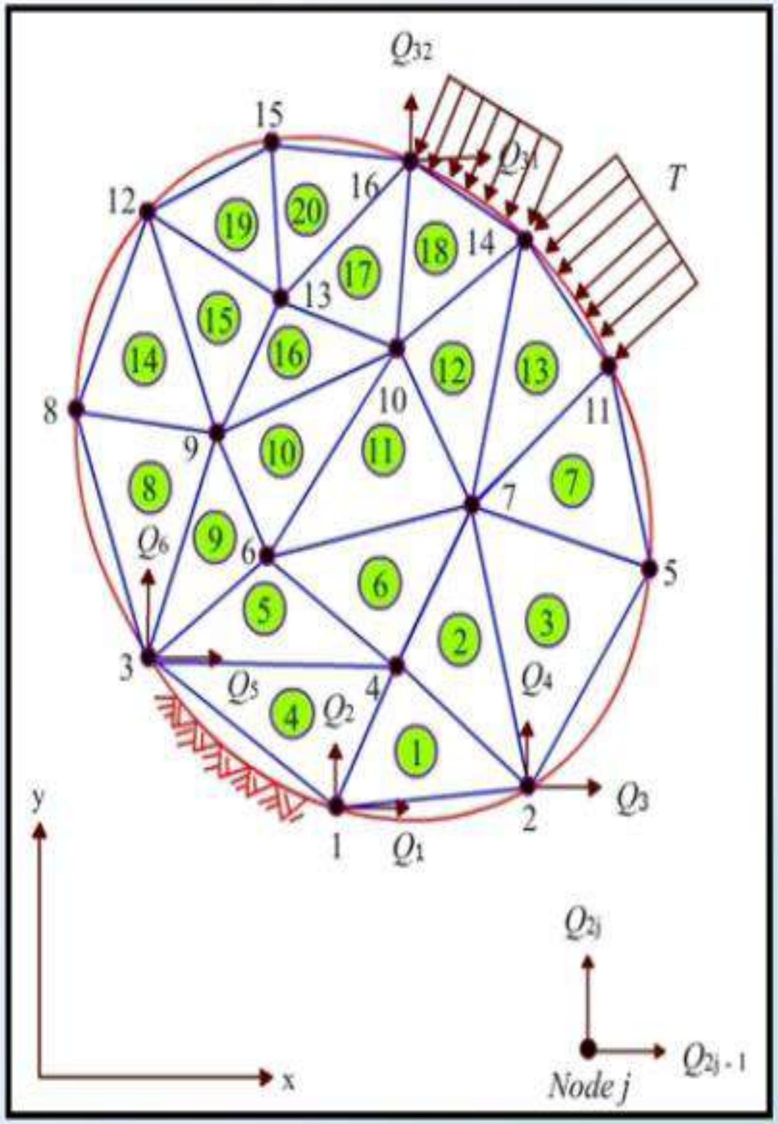


Fig.1 Finite Element Modeling

Finite Element Modeling

- The idea of the finite element method is to solve the continuous problem approximately, and this unfilled region contributes to some part of this approximation.
- In the two-dimensional problem discussed here, each node is permitted to displace in the two directions x and y. Thus, each node has two degrees of freedom (dof's).
- The displacement components of node j are taken as Q_{2j-1} in the x direction and Q_{2j} in y direction.
- We denote the global displacement vector as,

$$Q=[Q_1, Q_2, Q_3, \ldots\ldots Q_N]^T$$

where N is the number of degrees of freedom.

Finite Element Modeling

- Computationally, the information on the triangulation is to be represented in the form of nodal coordinates and connectivity.
- The nodal coordinates are stored in a two dimensional array represented by the total number of nodes and the two coordinates per node.
- Figure shows the nodal connectivity for a typical element,
- For the three nodes designated locally as 1,2, and 3, the corresponding global node numbers are defined in Fig.1.

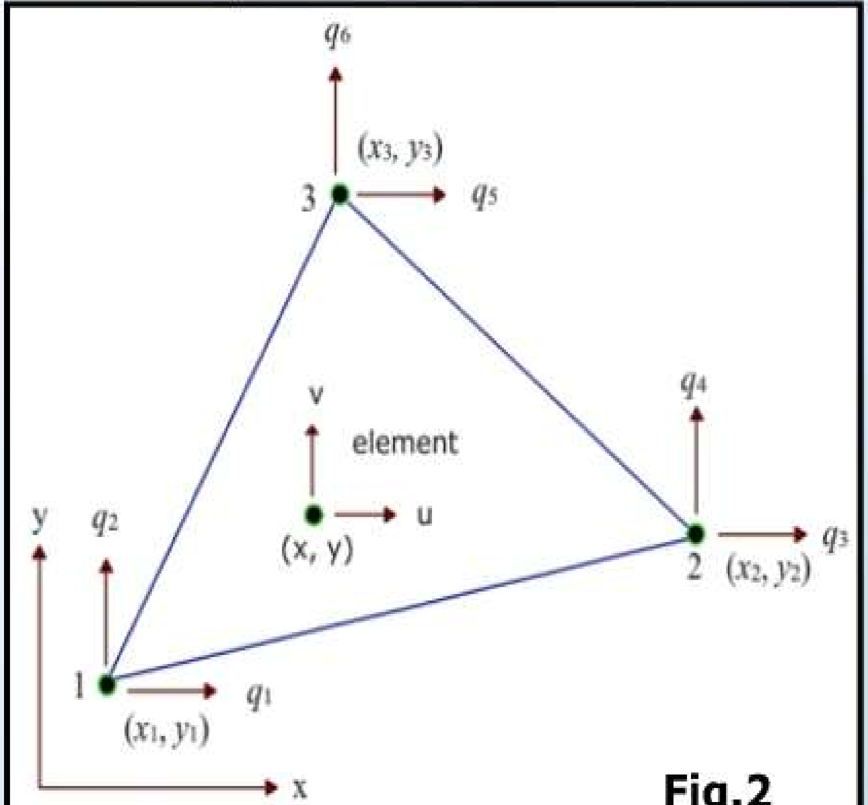


Fig.2

Finite Element Modeling

- This element connectivity information becomes an array of the size and number of elements and three nodes per element. A typical connectivity representation is shown in Table.1.

TABLE 1: Element connectivity

Element number e	Three nodes 1	2	3
1	1	2	4
2	4	2	7
⋮			
11	6	7	10
⋮			
20	13	16	15

- The nodal coordinates designated by (x_1,y_1), (x_2,y_2) *and* (x_3,y_3) have the global correspondence established through Table.1. The local representation of nodal coordinates and degrees of freedom provides a setting for a simple and clear representation of element characteristics

Constant Strain Triangle

- The displacements at points inside an element need to be represented in terms of the nodal displacements of the element.
- For the constant strain triangle, the shape functions are linear over the element. The three shape functions N_1 , N_2 , *and* N_3 corresponding to nodes 1,2, and 3, respectively, are shown in Figure.3 below.
- Shape function N_1 is 1 at node 1 and linearly reduces to 0 at nodes 2 and 3. The values of shape function N thus define a plane surface shown shaded in Figure.3(a)

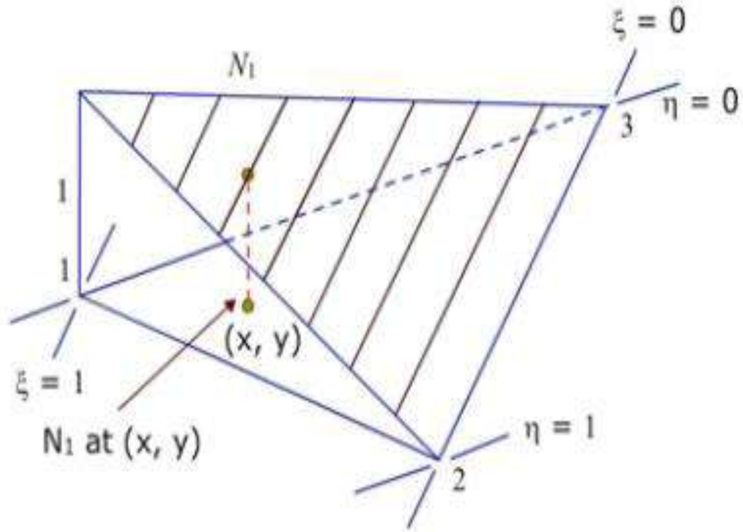
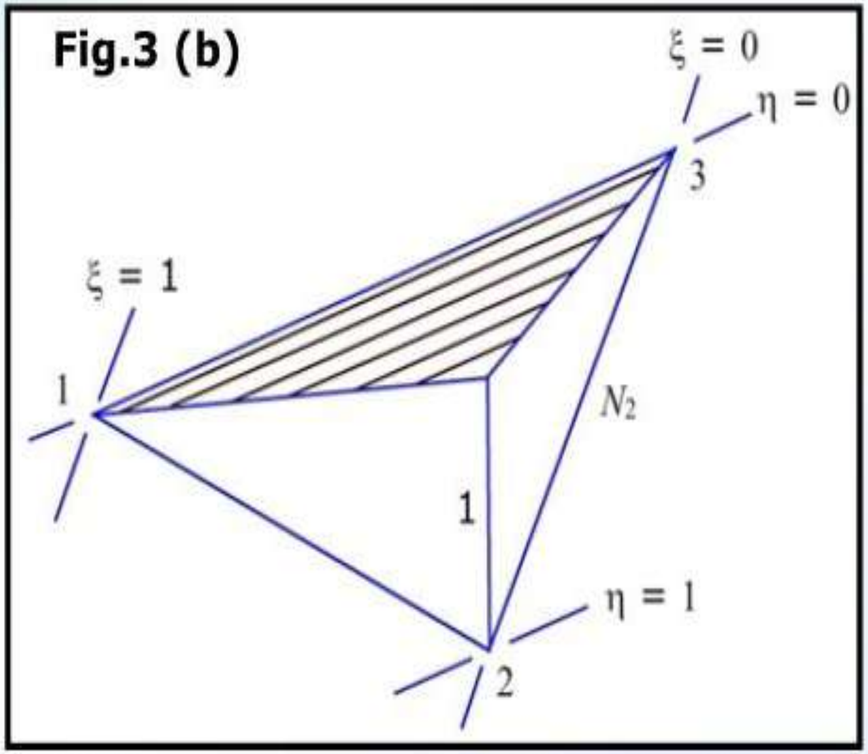
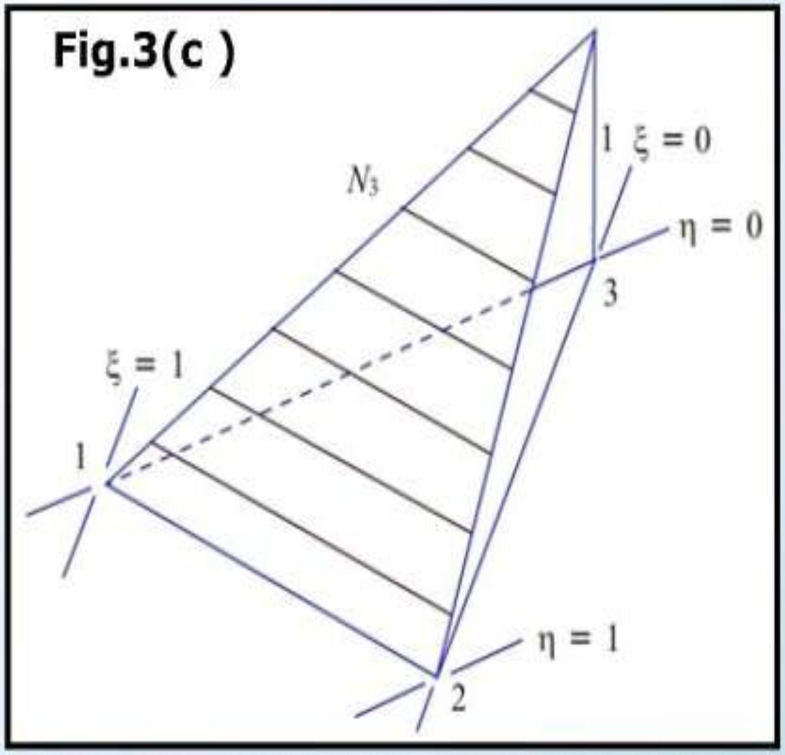


Fig 3(a)

Constant Strain Triangle

- Similarly N_2 and N_3 are represented by similar surfaces having values of 1 at nodes 2 and 3, respectively, and dropping to 0 at the opposite edges as shown in fig.3 (b) and (c).





- Any linear combination of these shape functions also represents a plane surface. In particular, $N_1 + N_2 + N_3$ represents a plane at a height of 1 at nodes 1, 2, and 3, and, thus, it is parallel to the triangle 123.

Constant Strain Triangle

- Consequently, for every N_1 , N_2, and N_3,

$$N_1 + N_2 + N_3 = 1$$

N_1 , N_2 , and N_3 are therefore not linearly independent; only two of these are independent.

- The independent shape functions are conveniently represented by the pair ξ, η as,

$$N_1 = \xi \qquad N_2 = \eta \qquad N_3 = 1 - \xi - \eta \quad \text{---- } 1$$

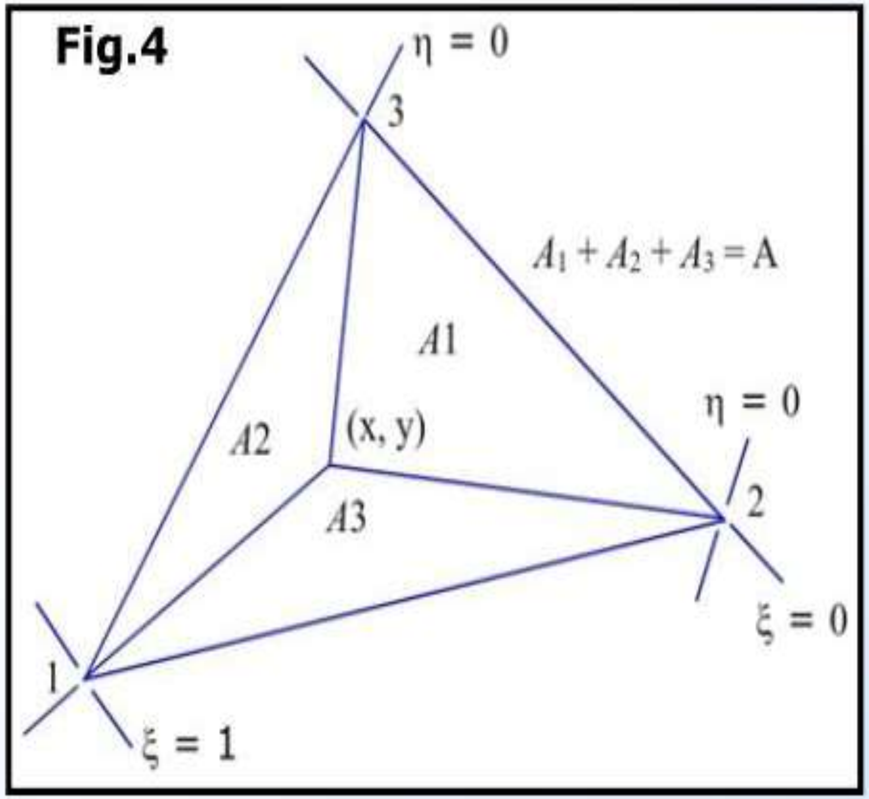
Where ξ, η are natural co-ordinates as shown in figure.3.

- Here, in the two-dimensional problem, the x , y coordinates are mapped onto the ξ, η coordinates, and shape functions are defined as functions of ξ, η .

Constant Strain Triangle

❖ The shape functions can be physically represented by area coordinates. A point (x, y) in a triangle divides it into three areas, A_1, A_2, and A_3 as shown in Fig.4.

Fig.4



❖ The shape functions N_1 , N_2, and N_3 are precisely represented by,

$$N_1 = \frac{A_1}{A} \;,\; N_1 = \frac{A_2}{A} \;,\; N_1 = \frac{A_3}{A} \qquad \rightarrow \mathbf{1}$$

Where A is the area of the element.

Iso-Parametric Representation

- In iso-parametric representation element geometry and displacement are represented using the same set of shape functions.
- The displacements inside the elements , are now written using the shape functions and the nodal values of the unknown displacement field.

$$\begin{aligned} u &= N_1 q_1 + N_2 q_3 + N_3 q_5 \\ v &= N_1 q_2 + N_2 q_4 + N_3 q_6 \end{aligned} \quad \rightarrow \mathbf{2}$$

- Equation 2 can further be modified using equation 1 as,

$$\begin{aligned} u &= (q_1 + q_5)\xi + (q_3 + q_5)\eta + q_5 \\ v &= (q_2 + q_6)\xi + (q_4 + q_6)\eta + q_6 \end{aligned} \quad \rightarrow \mathbf{3}$$

Iso-Parametric Representation

- The relations in equation 2 can be written in matrix form by defining a shape function matrix as,

$$N = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 \end{bmatrix} \qquad \rightarrow \mathbf{4}$$

- Which can simply be written as,

$$U = Nq \qquad \rightarrow \mathbf{5}$$

- For the triangular element, the coordinates X, Y can also be represented in terms of nodal coordinates using the same shape functions.
- This is Isoparametric representation. This approach lends to simplicity of development and retains the uniformity with other complex elements.

Iso-Parametric Representation

❖ We have,

$$\left.\begin{aligned} x &= N_1 x_1 + N_2 x_2 + N_3 x_3 \\ y &= N_1 y_1 + N_2 y_2 + N_3 y_3 \end{aligned}\right\} \rightarrow \mathbf{6}$$

❖ Which can also be written as ,

$$\left.\begin{aligned} x &= (x_1 - x_3)\xi + (x_2 - x_3)\eta + x_3 \\ y &= (y_1 - y_3)\xi + (y_2 - y_3)\eta + y_3 \end{aligned}\right\} \rightarrow \mathbf{7}$$

❖ Using the notations $x_{ij} = x_i - x_j$ and $y_{ij} = y_i - y_j$ in equation 7 we get ,

$$\left.\begin{aligned} x &= x_{13}\xi + x_{23}\eta + x_3 \\ y &= y_{13}\xi + y_{23}\eta + y_3 \end{aligned}\right\} \rightarrow \mathbf{8}$$

❖ Equation 8 relates x and y co-ordinates in-terms of ξ and η coordinates. Equation 3 relates u and v as functions of ξ and η.

Solved Problem

1. Evaluate the shape functions N_1, N_2 and N_3 at the interior point P for the triangular element shown in figure

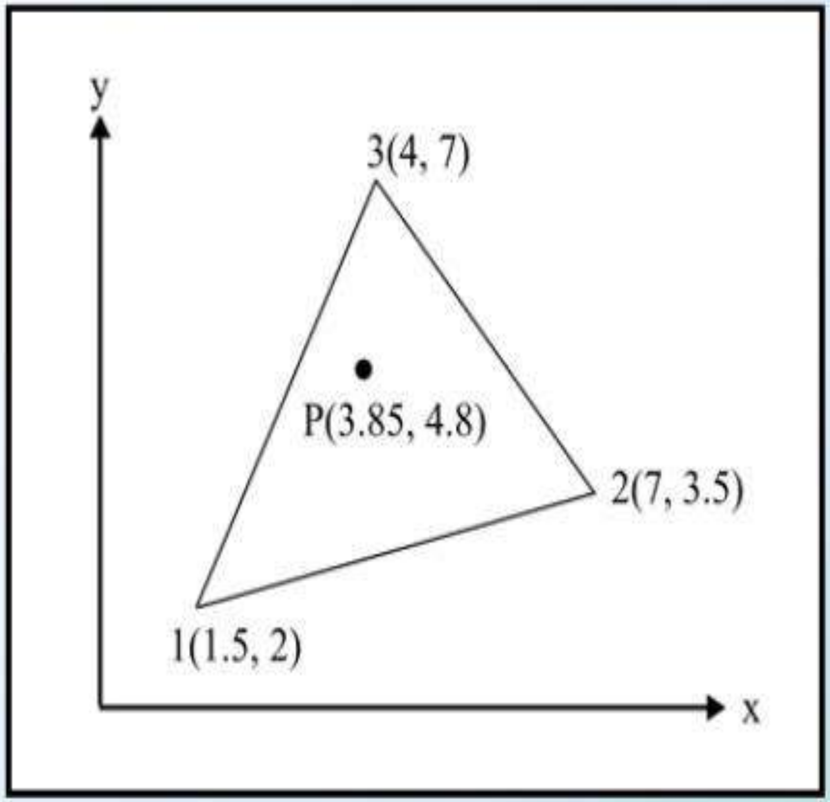
Given data :-

Coordinates at node 1 (x_1 , y_1) = (1.5, 2)

Co ordinates at node 2 (x_2 , y_2) = (7, 3.5)

Coordinates at node 3 (x_3 , y_3) = (4, 7)

Coordinates at interior point P = (3.85, 4.8)



Solved Problem

To find :-

Shape functions N_1, N_2 and N_3 at point P.

Solution : -

Using the isoparametric relations of equations 6,7,8 we have,

$$3.85 = 1.5N_1 + 7N_2 + 4N_3 = -2.5\xi + 3\eta + 4 \quad \rightarrow 1$$

$$4.8 = 2N_1 + 3.5N_2 + 7N_3 = -5\xi - 3.5\eta + 7 \quad \rightarrow 2$$

The above two equations are rearranged in the form,

$$2.5\xi - 3\eta = 0.15$$

$$5\xi + 3.5\eta = 2.2$$

On solving the equations we obtain,

$$\xi = 0.3 \quad \text{and} \quad \eta = 0.2$$

Solved Problem

Which implies that the values of shape functions are ,

$$N_1 = 0.3 \qquad N_2 = 0.2 \qquad N_3 = 0.5$$

Result :-

The values of shape functions at the interior point P are,

$$N_1 = 0.3 \qquad N_2 = 0.2 \qquad N_3 = 0.5$$

Jacobian of Transformation

- The Jacobian is defined as determinant of 2×2 matrix. In evaluating the strains, partial derivatives of u and v are to be taken with respect to x and y.
- From equations 2 to 8 , we see that ,

$$u = u(x(\xi,\eta), y(\xi,\eta)) \text{ and similarly } v = v(x(\xi,\eta), y(\xi,\eta))$$

- Using the chain rule for partial derivatives of u, we have,

$$\frac{\partial u}{\partial \xi} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \xi} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \xi}$$

$$\frac{\partial u}{\partial \eta} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \eta} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \eta}$$

which can be written in matrix notation as,

$$\begin{Bmatrix} \dfrac{\partial u}{\partial \xi} \\ \dfrac{\partial u}{\partial \eta} \end{Bmatrix} = \begin{bmatrix} \dfrac{\partial x}{\partial \xi} & \dfrac{\partial y}{\partial \xi} \\ \dfrac{\partial x}{\partial \eta} & \dfrac{\partial y}{\partial \eta} \end{bmatrix} \begin{Bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial u}{\partial y} \end{Bmatrix} \quad \rightarrow 9$$

Jacobian of Transformation

Where the (2 X 2) square matrix is denoted as the Jacobian of the transformation, J:

$$J = \begin{bmatrix} \dfrac{\partial x}{\partial \xi} & \dfrac{\partial y}{\partial \xi} \\ \dfrac{\partial x}{\partial \eta} & \dfrac{\partial y}{\partial \eta} \end{bmatrix} \quad \rightarrow \mathbf{10}$$

❖ On taking the derivative of x and y,

$$J = \begin{bmatrix} x_{13} & y_{13} \\ x_{23} & y_{23} \end{bmatrix} \quad \rightarrow \mathbf{11}$$

Jacobian of Transformation

Also from equation 9 ,

$$\begin{Bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial u}{\partial y} \end{Bmatrix} = J^{-1} \begin{Bmatrix} \dfrac{\partial u}{\partial \xi} \\ \dfrac{\partial u}{\partial \eta} \end{Bmatrix}$$

Where J^{-1} is the inverse of the Jacobian J. given by

$$J^{-1} = \frac{1}{\det J} \begin{bmatrix} y_{23} & -y_{13} \\ -x_{23} & x_{13} \end{bmatrix}$$

$$\det J = x_{13} y_{23} - x_{23} y_{13}$$

Jacobian of Transformation

- From the knowledge of the area of the triangle, it can be seen that the magnitude of det J is twice the area of the triangle. If the points 1,2, and 3 are ordered in a counterclockwise manner, det J is positive in sign. We have,

$$A = \frac{1}{2}\,|\det J\,|$$

where | | represents the magnitude.

Jacobian of Transformation

❖ From equations 10 and 11 it follows that,

$$\begin{Bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial u}{\partial y} \end{Bmatrix} = \frac{1}{\det J}\begin{bmatrix} y_{23}\dfrac{\partial u}{\partial \xi} & -y_{13}\dfrac{\partial u}{\partial \xi} \\ -x_{23}\dfrac{\partial u}{\partial \eta} & x_{13}\dfrac{\partial u}{\partial \eta} \end{bmatrix} \longrightarrow \text{12 a}$$

❖ Replacing u by the displacement v, we get a similar expression,

$$\begin{Bmatrix} \dfrac{\partial v}{\partial x} \\ \dfrac{\partial v}{\partial y} \end{Bmatrix} = \frac{1}{\det J}\begin{bmatrix} y_{23}\dfrac{\partial v}{\partial \xi} & -y_{13}\dfrac{\partial v}{\partial \xi} \\ -x_{23}\dfrac{\partial v}{\partial \eta} & x_{13}\dfrac{\partial v}{\partial \eta} \end{bmatrix} \longrightarrow \text{12 b}$$

Jacobian of Transformation

- Using the strain displacement equations, we get,

$$\in=\begin{Bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial v}{\partial y} \\ \dfrac{\partial u}{\partial y}+\dfrac{\partial v}{\partial x} \end{Bmatrix} \quad \rightarrow \textbf{13 a}$$

$$=\frac{1}{\det J}\begin{Bmatrix} y_{23}(q_1-q_5)-y_{13}(q_3-q_5) \\ -x_{23}(q_2-q_6)+x_{13}(q_4-q_6) \\ -x_{23}(q_1-q_5)+x_{13}(q_3-q_5)+y_{23}(q_2-q_6)-y_{13}(q_4-q_6) \end{Bmatrix}$$

- From the definition of we can write $y_{31}=y_{13}$ and $y_{12}=y_{13}-y_{23}$ and so on. The forgoing equation can be written of the form,

$$=\frac{1}{\det J}\begin{Bmatrix} y_{23}q_1-y_{31}q_3+y_{12}q_5 \\ x_{32}q_2+x_{13}q_4+x_{21}q_6 \\ x_{32}q_1+y_{23}q_2+x_{13}q_3+y_{31}q_4+y_{21}q_5+y_{12}q_6 \end{Bmatrix} \quad \rightarrow \textbf{13 b}$$

Jacobian of Transformation

- This equation can be written in matrix form as,

$$\epsilon = Bq \qquad \rightarrow \textbf{14}$$

where B is a (3 x 6) element strain-displacement matrix relating the three strains to the six nodal displacements and is given by,

$$B = \frac{1}{\det J}\begin{bmatrix} y_{23} & 0 & y_{31} & 0 & y_{12} & 0 \\ 0 & x_{32} & 0 & x_{13} & 0 & x_{21} \\ x_{32} & y_{23} & x_{13} & y_{31} & x_{21} & y_{12} \end{bmatrix} \qquad \rightarrow \textbf{15.a}$$

(or)

$$[B] = \frac{1}{2A}\begin{bmatrix} y_{23} & 0 & -y_{31} & 0 & y_{21} & 0 \\ 0 & -x_{32} & 0 & x_{31} & 0 & -x_{21} \\ -x_{32} & y_{32} & x_{31} & -y_{31} & -x_{21} & y_{21} \end{bmatrix} \qquad \rightarrow \textbf{15.b}$$

- It must be noted that all the elements of the B matrix are constants expressed in terms of the nodal coordinates.

Solved Problem

2) . Determine the Jacobian of the transformation J , for the triangular element shown.

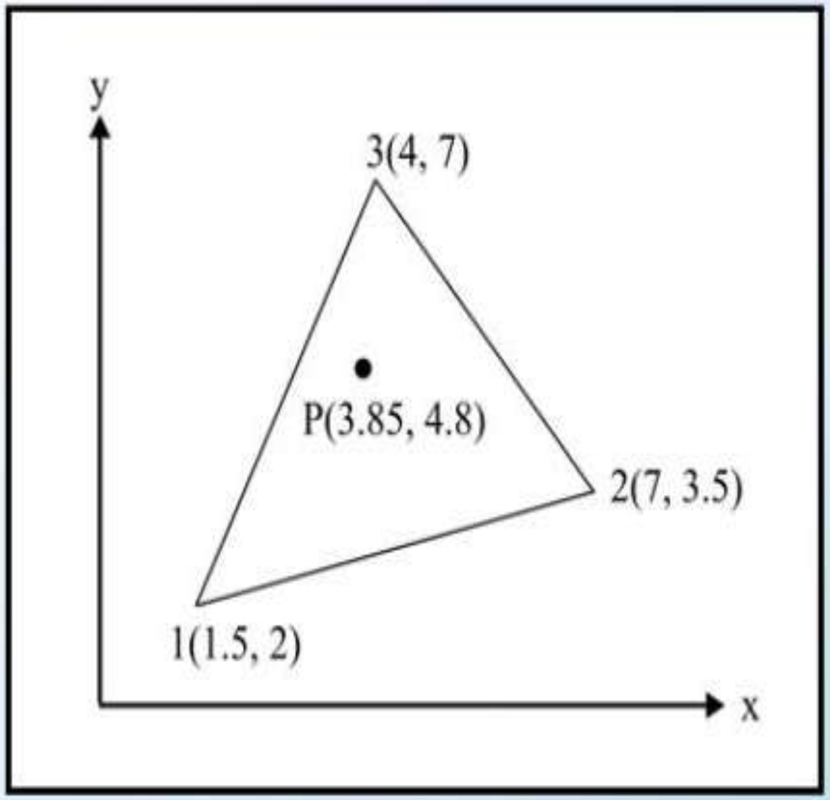
Given data :-

Coordinates at node 1 (x_1 , y_1) = (1.5 , 2)

Coordinates at node 2 (x_2 , y_2) = (7, 3.5)

Coordinates at node 3 (x_3 , y_3) = (4 , 7)

Coordinates at interior point P = (3.85 , 4.8)



To Find:-

Jacobian of Transformation J.

Solved Problem

Solution : -

We have,

$$J = \begin{bmatrix} x_{13} & y_{13} \\ x_{23} & y_{23} \end{bmatrix} = \begin{bmatrix} -2.5 & -5.0 \\ 3.0 & -3.5 \end{bmatrix}$$

Thus $det\, J$ = 23.75 units.

This is twice the area of triangle.

Result : -

Jacobian of Transformation for the given triangular element is 23.75 units.

Potential Energy Approach

- The potential energy of the system, $\prod$, is given by,

$$\prod = \frac{1}{2}\int_A \in^T D \in t dA - \int_A u^T f t \ dA - \int_L u^T t \ dl - \sum_i u_i^T P_i \qquad \rightarrow 16$$

- In the last term in equation 16 , 'I' indicates the point of application of load P_i and $P_i = [P_x, P_y]_i^T$
- The summation in '*i*' gives the potential energy due to all point loads.
- The total potential energy can be written in the form,

$$\prod = \sum_e \frac{1}{2}\int_A \in^T D \in t dA - \sum_e \int_A u^T f t dA - \int_L u^T t dl - \sum_i u_i^T P_i \qquad \rightarrow 17\text{ a}$$

or

$$\prod = \sum_e U_e - \sum_e \int_A u^T f t dA - \int_L u^T t dl - \sum_i u_i^T P_i \qquad \rightarrow 17\text{ b}$$

Where $U_e = \frac{1}{2}\int_A \in^T D \in t \, dA$ is the element strain energy.

Element Stiffness

❖ We now substitute for the strain from the element strain – displacement relationship in equation 14 into the element strain energy equation, we get ,

$$U_e = \frac{1}{2}\int_A \in^T D \in t\, dA$$

$$= \frac{1}{2}\int_A q^T B^T DBqtdA \qquad \rightarrow 18\text{ a}$$

❖ On solving we get ,

$$U_e = \frac{1}{2} q^T k^e q \qquad \rightarrow 18\text{ b}$$

Where, K^e is the element stiffness matrix given by

$$k^e = t_e A_e B^T DB \qquad \rightarrow 18\text{ c}$$

Force Terms

- The body force term $\int_e u^T ftdA$ appearing in the total potential energy in equation 17.b is considered first . We have ,

$$\int_e u^T ftdA = t_e \int_e \left(uf_x + vf_y \right) dA \qquad \rightarrow 19a.$$

- Using the interpolation relations, we get,

$$\begin{aligned} \int_e u^T ftdA = {} & q_1 t\left(t_e f_x \int_e N_1 dA\right) + q_2\left(t_e f_y \int_e N_2 dA\right) \\ & + q_3\left(t_e f_x \int_e N_2 dA\right) + q_4\left(t_e f_y \int_e N_2 dA\right) \qquad \rightarrow 19.b \\ & + q_5 t\left(t_e f_x \int_e N_3 dA\right) + q_6\left(t_e f_y \int_e N_3 dA\right) \end{aligned}$$

Force Terms

- $\int_e N_i \, dA$ Represents the volume of a tetrahedron with base area A_e and height of corner equal to 1 (non-dimensional).

- The volume of this tetrahedron is given by $\{(1/3)$ x Base area x Height$\}$.

  Hence, We have,

$$\int_e N_i \, dA = \frac{1}{3} Ae$$

- Equation 19.b can now be written in the form, $\int_e U^T ft \, dA = q^T f^e$

  Where, f^e is the element body force vector given by,

$$= \frac{t_e A_e}{3} [f_x f_y \; f_x f_y \; f_x f_y]^T$$

Force Terms

- A traction force is a distributed load acting on the surface of the body. Such a force acts on edges connecting boundary nodes. A traction force acting on the edge of an element contributes to the global load vector F.
- The point load term is easily considered by having a node at the point of application of the point load.
- The contribution of body forces, traction forces, and point loads to the global force F can be represented as,

$$\mathrm{F} \leftarrow \sum_e (f^e + T^e) + \mathrm{P}$$

Stress calculations:-

- Since strains are constant in a constant-strain triangle (CST) element, the corresponding stresses are constant.

Force Terms

- The stress values need to be calculated for each element. Using the stress-strain relations, we have,

$$\sigma = DBq \qquad \rightarrow 20$$

- Using the element connectivity table we now extract the element nodal displacements q from the global displacements vector Q.
- If the element is in a state of plane stress, the stress-strain relations are given by

$$\vec{\sigma} = \left|D\right|\vec{\varepsilon} \qquad \rightarrow 21$$

Where

$$\vec{\sigma} = \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{Bmatrix}$$

- Equation 20 is used to calculate the elemental stresses. Principal stresses and their directions are calculated using Mohr's circle relationships.

Characteristics of Constant Strain Triangle Element

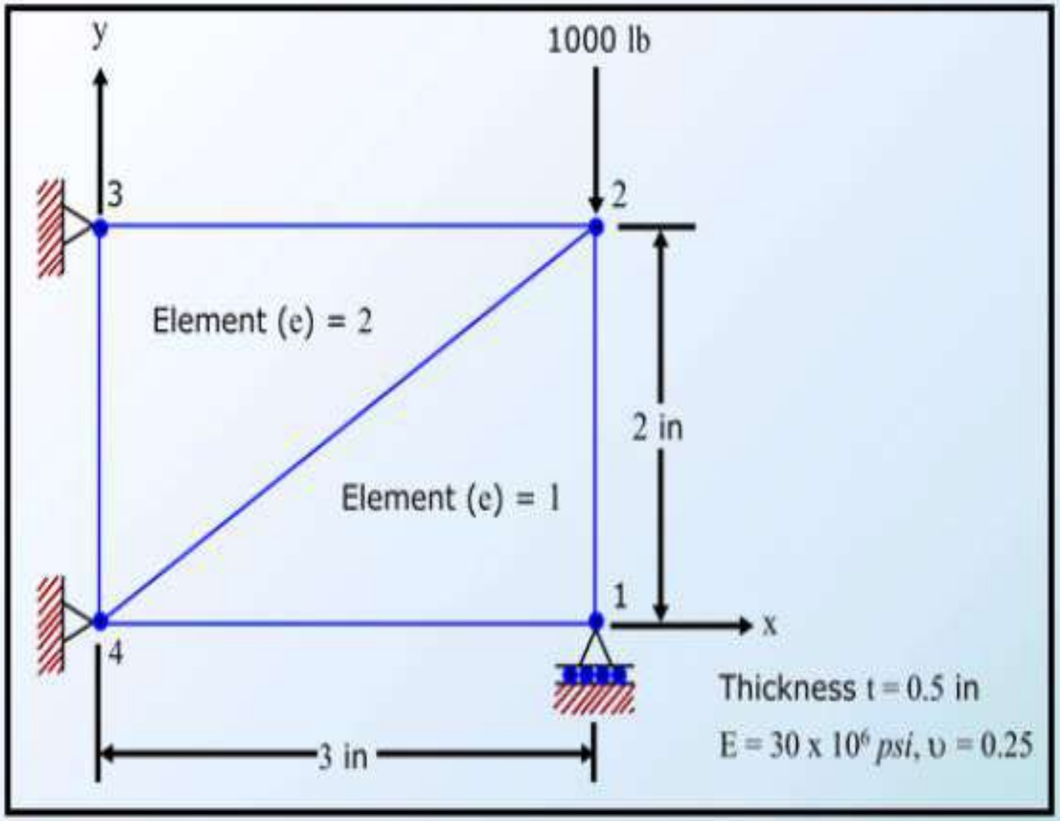
- The CST element was the first finite element developed for the analysis of plane stress problems .
- Because the displacement model is linear (Equation.2), the element is called a linear triangular element. From Equations (15.a) and (15.b), we find that the *[B]* matrix is independent of the position within the element and hence the strains are constant throughout the element.
- This is the reason why this element is often referred to as a CST element (constant strain triangular element).
- The displacement model chosen (Equation.2) guarantees continuity of displacements with adjacent elements because the displacements vary linearly along any side of the triangle (due to linear model).

Characteristics of Constant Strain Triangle Element

- From Equation.21, we can notice that the stresses are also constant inside an element.
- Hence, the element is also called a CST (Constant Stress Triangular) element.
- More accurate normal stresses can be obtained by using smaller size elements.
- However, the convergence to the correct solution will be very slow.

Solved Problems

3. For the two-dimensional loaded plate in figure determine the displacements of nodes 1 and 2 and the element stress using plane stress conditions. Body force may be neglected in comparison with the external force



Given Data :

Thickness (t) = 0.5 inches.

Modulus of elasticity (E) = 30 x 10^6 psi, Poisson's Ratio ν = 0.25

Displacements Q_2 (acting in y direction at node1), Q_5, Q_6 (acting in x, y directions at node 3), Q_7, Q_8 (acting in x and y direction at node 4) are all zero.

Solved Problems

To Find:-

The displacement and stresses at nodes 1 and 2.

Solution:-

For plane stress conditions, the material property matrix is given by,

$$D=\frac{E}{1-v}\begin{bmatrix} 1 & v & 0 \\ v & 1 & 0 \\ 0 & 0 & \frac{1-v}{2} \end{bmatrix}=\begin{bmatrix} 3.2\times10^{7} & 0.8\times10^{7} & 0 \\ 0.8\times10^{7} & 3.2\times10^{7} & 0 \\ 0 & 0 & 1.2\times10^{7} \end{bmatrix}$$

We establish the connectivity from the figure as follows,

Element No	Nodes		
	1	2	3
1	1	2	4
2	3	4	4

Solved Problems

On performing the matrix multiplication DB^e, we get,

$$DB^1 = 10^7 \begin{bmatrix} 1.067 & -0.4 & 0 & 0.4 & -1.067 & 0 \\ 0.267 & -1.6 & 0 & 1.6 & -0.267 & 0 \\ -0.6 & 0.4 & 0.6 & 0 & 0 & -0.4 \end{bmatrix}$$

$$DB^2 = 10^7 \begin{bmatrix} -1.067 & 0.4 & 0 & -0.4 & 1.067 & 0 \\ -0.267 & 1.6 & 0 & -1.6 & 0.267 & 0 \\ -0.6 & -0.4 & -0.6 & 0 & 0 & 0.4 \end{bmatrix}$$

These two relationships will be used later in calculating stresses using, $\sigma^e = DB^e q$

The multiplication, $t_e A_e B^{eT} DB^e$ gives the element stiffness matrices .

Solved Problems

The element stiffness matrices are,

$$k^1 = 10^7 \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 7 & 8 \end{array} \leftarrow \text{Global dof} \\ \begin{bmatrix} 0.983 & -0.5 & -0.45 & 0.2 & -0.533 & 0.3 \\ -0.5 & 1.4 & 0.3 & -1.2 & 0.2 & -0.2 \\ -0.45 & 0.3 & 0.45 & 0 & 0 & -0.3 \\ 0.2 & -1.2 & 0 & 12 & -0.2 & 0 \\ -0.533 & 0.2 & 0 & -0.2 & 0.533 & 0 \\ 0.3 & -0.2 & -0.3 & 0 & 0 & 0.2 \end{bmatrix} \end{array}$$

$$k^2 = 10^7 \begin{array}{c} \begin{array}{cccccc} 5 & 6 & 7 & 8 & 3 & 4 \end{array} \leftarrow \text{Global dof} \\ \begin{bmatrix} 0.983 & -0.5 & -0.45 & 0.2 & -0.533 & 0.3 \\ -0.5 & 1.4 & 0.3 & -1.2 & 0.2 & -0.2 \\ -0.45 & 0.3 & 0.45 & 0 & 0 & -0.3 \\ 0.2 & -1.2 & 0 & 12 & -0.2 & 0 \\ -0.533 & 0.2 & 0 & -0.2 & 0.533 & 0 \\ 0.3 & -0.2 & -0.3 & 0 & 0 & 0.2 \end{bmatrix} \end{array}$$

Solved Problems

Now its sufficient to consider the stiffness associated with Q_1, Q_3 and Q_4.

Since the body forces are neglected, the first vector has the component F_4 = -1000lb.

(where lb is unit of pounds , 1 lb = 0.453 Kg).

The set of equations is given by the matrix representation,

$$10^7 \begin{bmatrix} 0.983 & -0.45 & 0.2 \\ -0.45 & 0.983 & 0 \\ 0.2 & 0 & 1.4 \end{bmatrix} \begin{Bmatrix} Q_1 \\ Q_3 \\ Q_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ -1000 \end{Bmatrix}$$

Performing matrix multiplication and solving for the displacements we get,

Q_1= 1.913 x 10^{-5} in. Q_3=0.875x10^{-5}in. Q_4=-7.436x10^{-5}in.

Solved Problems

For element 1, the element nodal displacement vector is given by,

$$q^1 = 10^{-5}\left[1.913, 0, 0.875 - 7.436, 0, 0\right]^T$$

The element stresses σ^1 are calculated from DB^1q as

$$\sigma^1 = \left[-9.33, -1138.7, -62.3\right]^T \ psi$$

Similarly,

$$q^2 = 10^{-5}\left[0, 0, 0, 0, 0.875, -7.436\right]^T$$

$$\sigma^2 = \left[93.4, 23.4, -297.4\right]^T \ psi$$

Solved Problems

Result:-

The nodal displacements Q_1, Q_3 and Q_4 are calculated as ,

Q_1= 1.913x10^{-5}in.

Q_3=0.875x10^{-5}in.

Q_4=-7.436x10^{-5}in.

The nodal displacement vector and element stresses for elements 1 and 2 are calculated as,

$$q^1 = 10^{-5}\left[1.913, 0, 0.875 - 7.436, 0, 0\right]^T$$

$$\sigma^1 = \left[-9.33, -1138.7, -62.3\right]^T psi$$

$$q^2 = 10^{-5}\left[0, 0, 0, 0, 0.875, -7.436\right]^T$$

$$\sigma^2 = \left[93.4, 23.4, -297.4\right]^T psi$$

Treatment of Boundary Conditions

- The finite element method is used for computing displacements and stresses for a wide variety of problems.
- The physical dimensions, loading, and boundary conditions are clearly defined in the problems we discussed .
- In other problems, these are not clear at the outset.

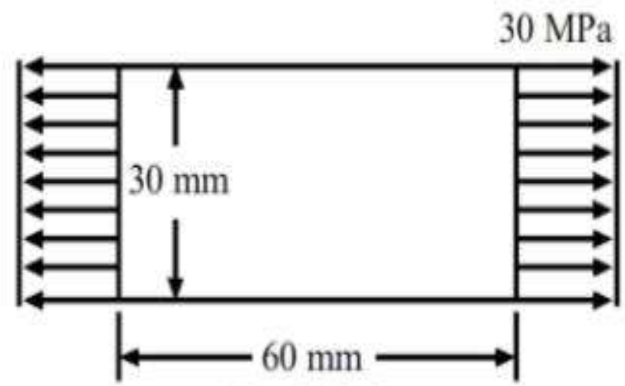


Fig 5.8(a)

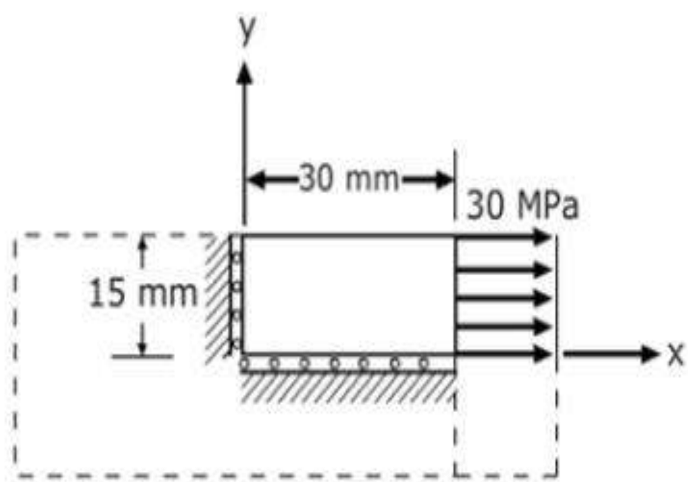


Fig 5.8(b)

Rectangular plate

Treatment of Boundary Conditions

- An example is the problem illustrated in Fig. 5.8a.
- A plate with such a loading can exist anywhere in space. Since we are interested in the deformation of the body, the symmetry of the geometry and the symmetry of the loading can be used effectively.
- Let x and y represent the axes of symmetry as shown in Fig. 5.8b. The points along the x-axis move along x and are constrained in the y direction and points along the y-axis are constrained along the x direction.
- This suggests that the part, which is one-quarter of the full area, with the loading and boundary conditions as shown is all that is needed to solve the deformation and stresses

Two Dimensional Stress Analysis - II

Learning Objectives

At the end of this topic you will be able to:

- ❖ Know about axisymmetric Solids
- ❖ Derive the relationship between stress and strain
- ❖ Formulate axisymmteric problems

Outcomes

By the end of this topic, you will be able to:

- ❖ Understand about axisymmetric Solids
- ❖ Differentiate the relationship between stress and strain
- ❖ Solve the problems relating axisymmteric

Axisymmetric Solids

- Axisymmteric solids are solids with axial symmetry.
- These solids have their geometrical and material properties independent of the circumferential co-ordinate, θ. As shown in figure 1 .

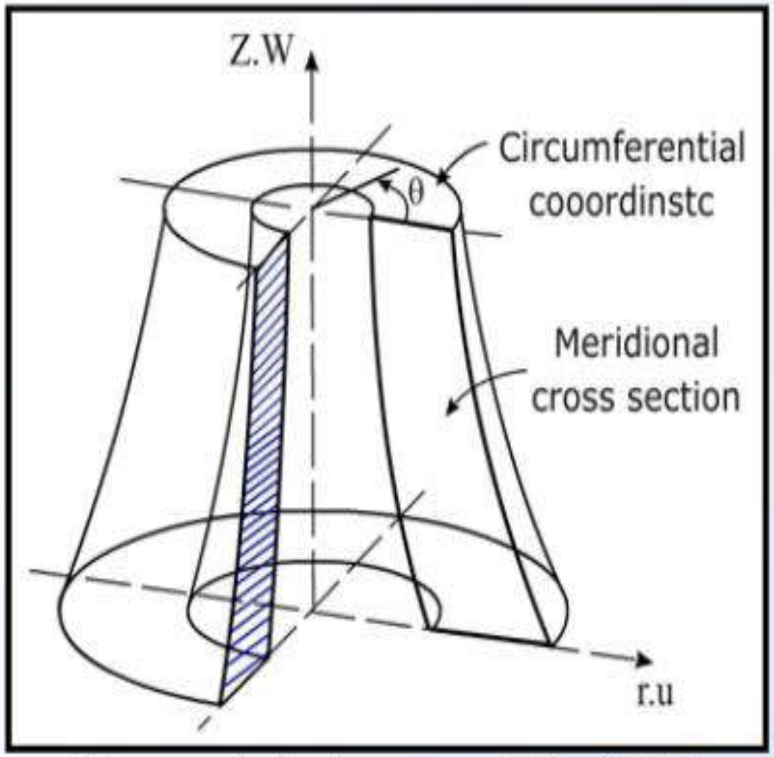


Figure 1 Axisymmetric Solid

- Few examples of axisymmetric solids are water reservoir, water treatment plant , cooling tower , cylindrical tank with spherical dome as shown in fig.2.

Axisymmetric Solids

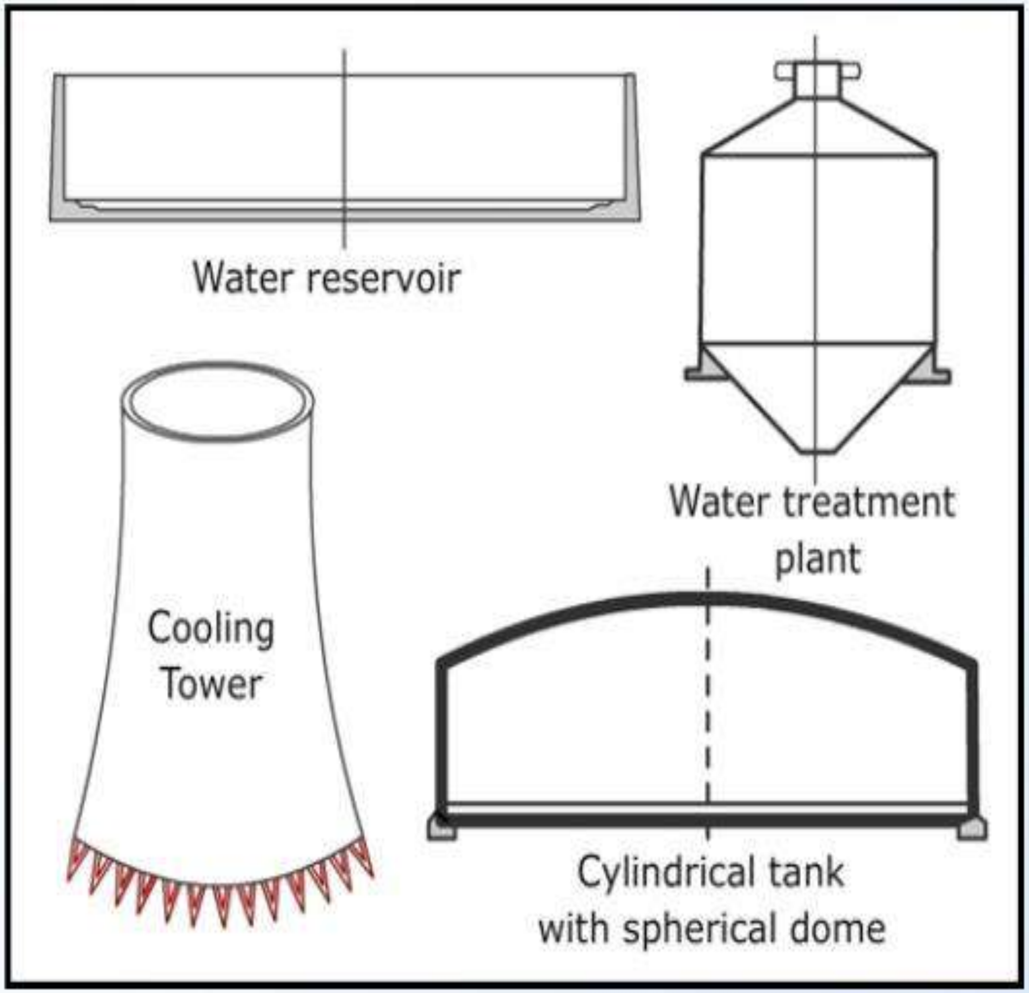


Figure 2 Examples of Axisymmetric Solids

- Problems involving three-dimensional axisymmetric solids or solids of revolution, subjected to axisymmetric loading, reduce to simple two-dimensional problems.

Axisymmetric Solids

❖ Because of total symmetry about the z-axis, as seen in Fig.3.a, all deformations and stresses are independent of the rotational angle θ.

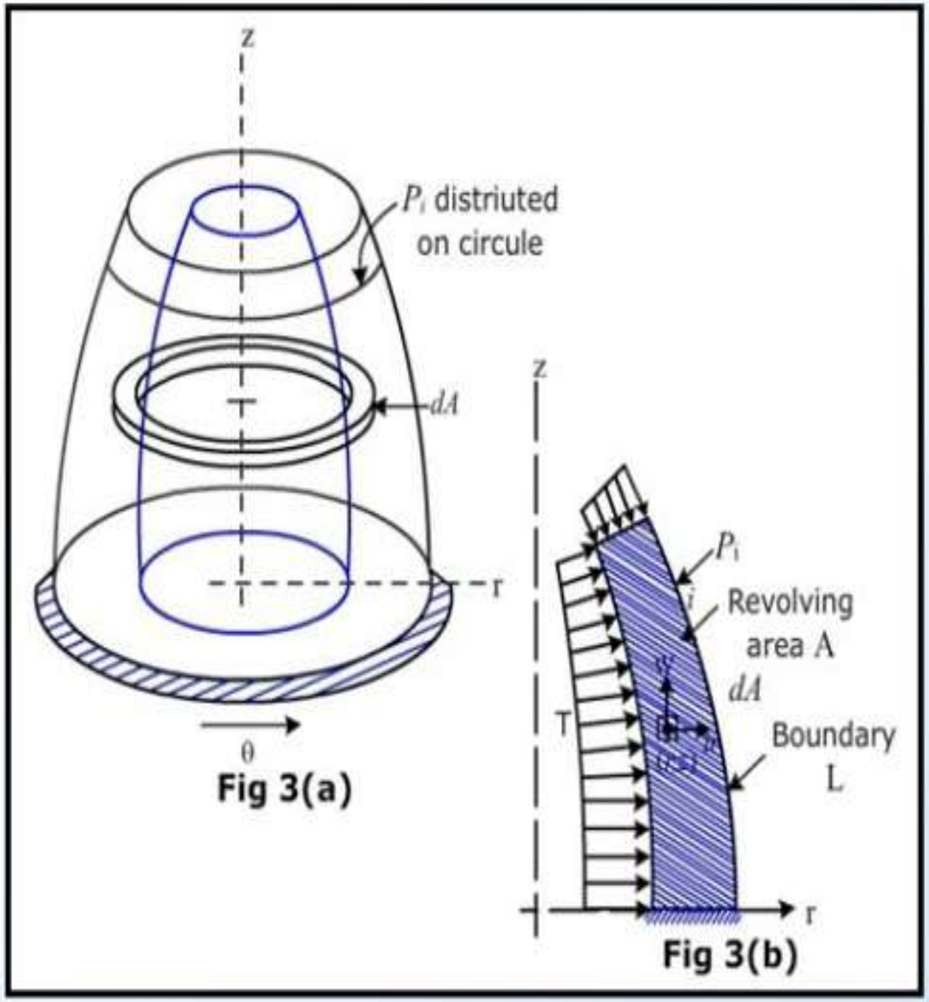


Figure 3

Axisymmetric Solids

- Thus, the problem needs to be looked at as a two dimensional problem in rz, defined on the revolving area (Fig.3.b).
- Gravity forces can be considered if acting in the z direction. Revolving bodies like flywheels can be analyzed by introducing centrifugal forces in the body force term.
- The various forces acting on Figure.3.b are displacement force term, body force term, traction force, point load, written as,

$$u = [u, w]^T$$

$$T = [T_r, T_z]^T$$

$$f = [f_r, f_z]^T$$

$$P = [P_r, P_z]^T$$

Axisymmetric Formulation

❖ Considering the elemental volume shown in Fig.4, the potential energy can be written in the form,

$$\Pi = \frac{1}{2}\int_0^{2\pi}\int_A \sigma^T \in r dA d\theta - \int_0^{2\pi}\int_A u^T f r dA d\theta - \int_0^{2\pi}\int_L u^T T r dl d\theta - \sum_i u_i^T P_i \qquad \rightarrow 1$$

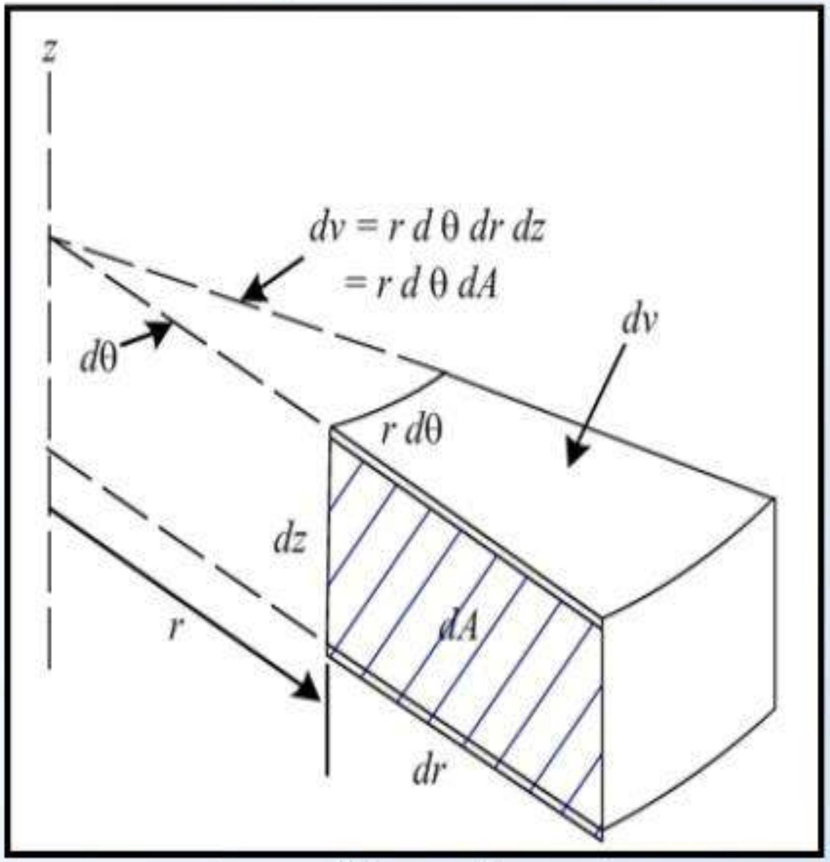


Figure 4

Axisymmetric Formulation

- Where, $rdld\theta$ is the elemental surface area and the point load P_i represents a line load distributed around a circle, as shown in Fig.3.b.
- All variables in the integrals are independent of θ. Thus, Eq. 1 can be written as,

$$\Pi = 2\pi \frac{1}{2}\int_A \sigma^T \in rdA - \int_A u^T frdA - \int_A u^T Trdl - \sum_i u_i^T P_i \qquad \rightarrow 2$$

Where,

$$u = [u, w]^T \qquad \rightarrow 3$$

$$f = [f_r, f_z]^T \qquad \rightarrow 4$$

$$T = [T_r, T_z]^T \qquad \rightarrow 5$$

Axisymmetric Formulation

❖ From Fig.5, we can write the relationship between strains $\boldsymbol{E}$ and displacements $\boldsymbol{u}$ as,

$$\epsilon = \left[\epsilon_r, \epsilon_z, \gamma_{rz}, \epsilon_\theta\right]^T$$

$$= \left[\frac{\partial u}{\partial r}, \frac{\partial w}{\partial z}, \frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}, \frac{u}{r}\right]^T \quad \rightarrow 6$$

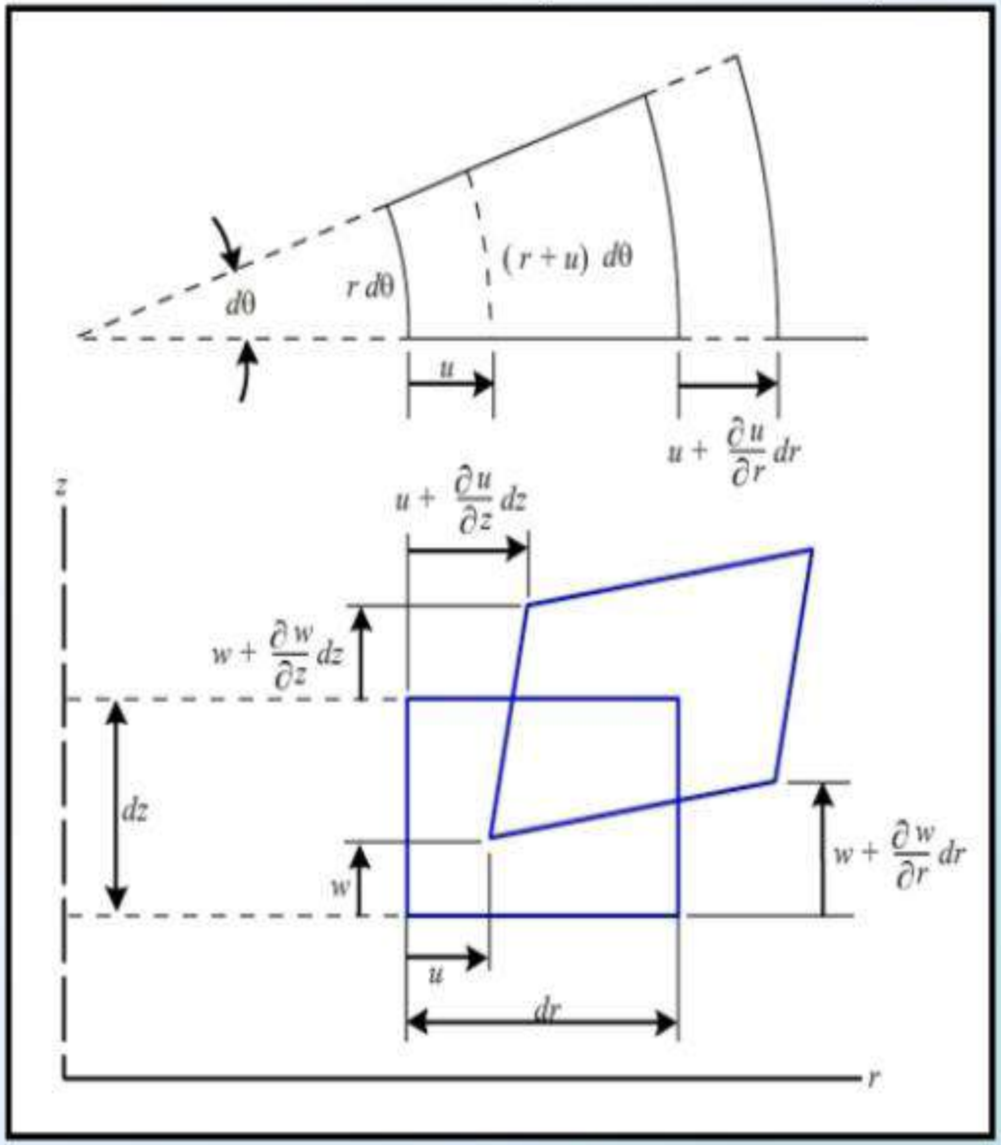


Figure 5

Axisymmetric Formulation

❖ The stress vector is correspondingly defined as,

$$\sigma = \left[\sigma_r, \sigma_z, \sigma_{rz}, \sigma_\theta\right]^T \quad \rightarrow 7$$

❖ The stress-strain relations are given in the usual form, viz.,

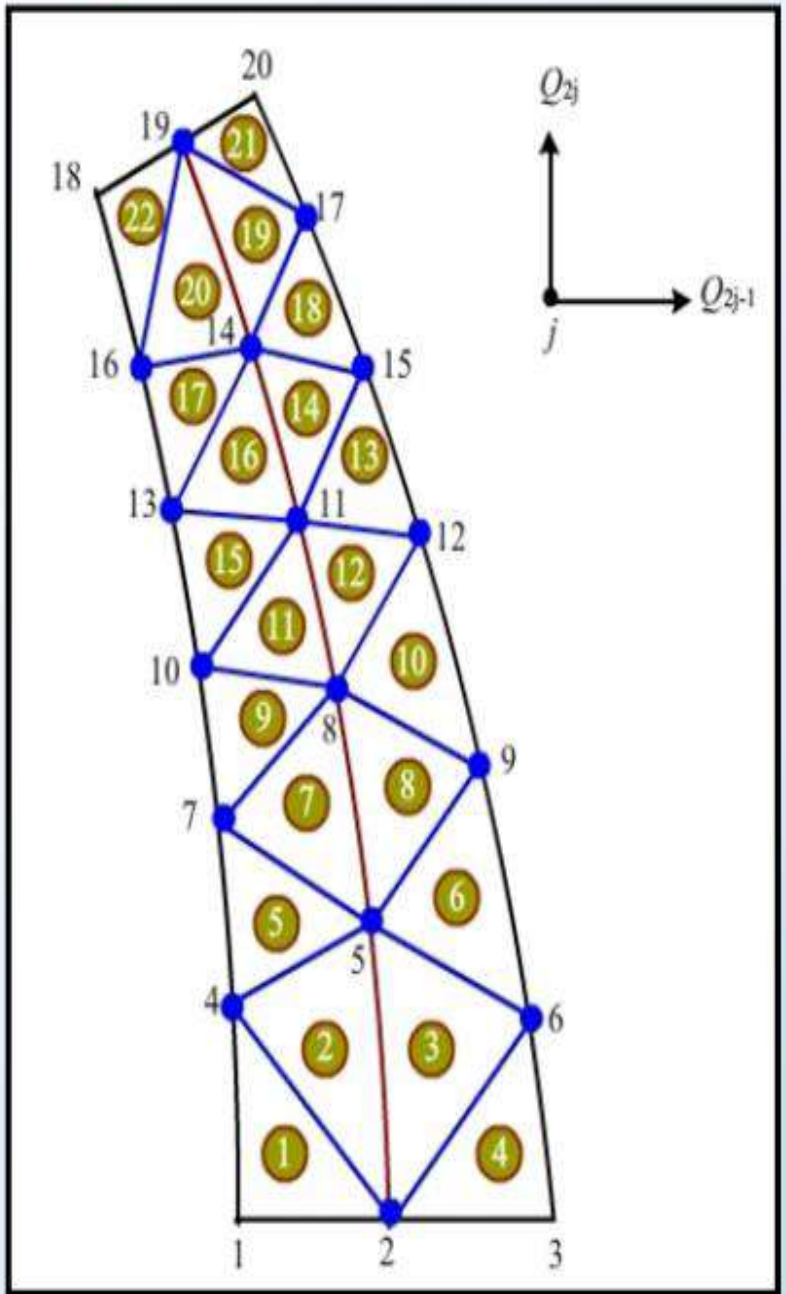
$$\sigma = D\epsilon \quad \rightarrow 8$$

where, the (4 x 4) matrix D can be written as ,

$$D = \frac{E(1-v)}{(1+v)(1-2v)} \begin{bmatrix} 1 & \frac{v}{1-v} & 0 & \frac{v}{1-v} \\ \frac{v}{1-v} & 1 & 0 & \frac{v}{1-v} \\ 0 & 0 & \frac{1-v}{2(1-v)} & 0 \\ \frac{v}{1-v} & \frac{v}{1-v} & 0 & 1 \end{bmatrix} \quad \rightarrow 9$$

Finite Element Modeling – Triangular Element

- The two-dimensional region defined by the revolving area is divided into triangular elements, as shown in Fig.6.
- Though each element is completely represented by the area in the rz plane, in reality, it is a ring-shaped solid of revolution obtained by revolving the triangle about the z-axis.



Finite Element Modeling – Triangular Element

❖ A typical element is shown in Fig.7.

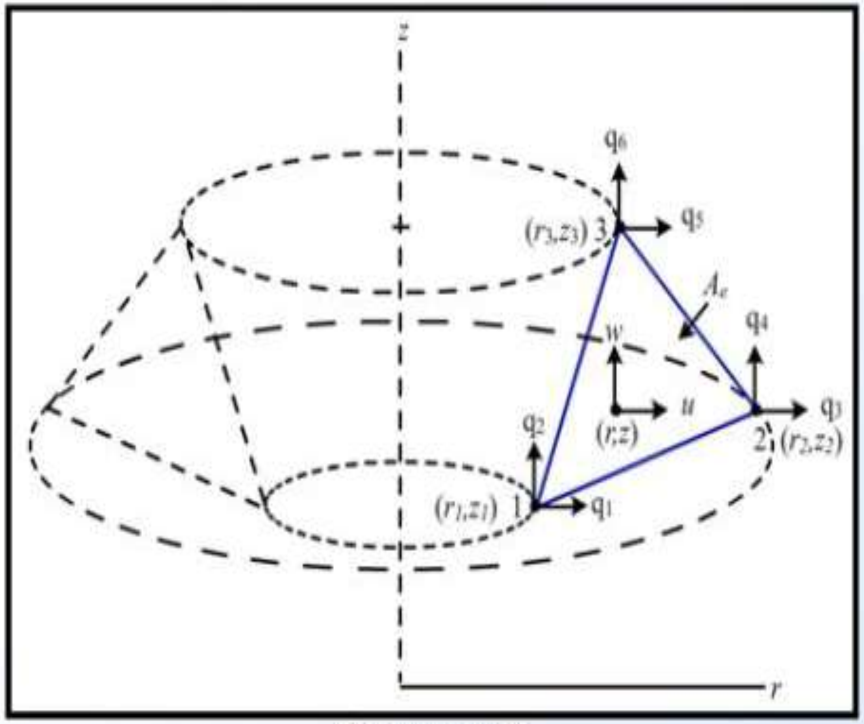


Figure 7

❖ The definition of connectivity of elements and the nodal coordinates follow the steps involved in the CST element.

❖ We note here that the r- and z- coordinates, respectively, replace x and y.

❖ Using the three shape functions N_1, N_2, and N_3 . we define,

$$u = Nq \qquad \rightarrow 10$$

Finite Element Modeling – Triangular Element

- Where u is defined in equation 3 and,

$$N = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 \end{bmatrix} \quad \rightarrow 11$$

$$q = \left[q_1, q_2, q_3, q_4, q_5, q_6\right]^T \quad \rightarrow 12$$

- If we denote $N_1 = \xi$ *and* $N_1 = \eta$ *, and note that* $N_3 = 1 - \xi - \eta$ *, then Equation 10 gives,*

$$\begin{aligned} u &= \xi q_1 + \eta q_3 + (1-\xi-\eta) q_5 \\ \omega &= \xi q_2 + \eta q_4 + (1-\xi-\eta) q_6 \end{aligned} \quad \rightarrow 13$$

- By using the isoparametric representation, we find,

$$\begin{aligned} r &= \xi r_1 + \eta r_3 + (1-\xi-\eta) r_5 \\ z &= \xi z_1 + \eta z_3 + (1-\xi-\eta) z_5 \end{aligned}$$

$$\rightarrow 14$$

Finite Element Modeling – Triangular Element

❖ The chain rule of differentiation gives,

$$\begin{Bmatrix} \dfrac{\partial u}{\partial \xi} \\ \dfrac{\partial u}{\partial \eta} \end{Bmatrix} = J \begin{Bmatrix} \dfrac{\partial u}{\partial r} \\ \dfrac{\partial u}{\partial z} \end{Bmatrix} \qquad \rightarrow 15$$

$$\begin{Bmatrix} \dfrac{\partial \omega}{\partial \xi} \\ \dfrac{\partial \omega}{\partial \eta} \end{Bmatrix} = J \begin{Bmatrix} \dfrac{\partial \omega}{\partial r} \\ \dfrac{\partial \omega}{\partial z} \end{Bmatrix} \qquad \rightarrow 16$$

where, the Jacobian is given by ,

$$J = \begin{pmatrix} r_{13} & z_{13} \\ r_{23} & z_{23} \end{pmatrix} \qquad \rightarrow 17a$$

Finite Element Modeling – Triangular Element

❖ Replacing the nodes as i,j,k instead of 1,2,3 , we get the relation as,

$$\begin{Bmatrix} N_i \\ N_j \\ N_k \end{Bmatrix} = \begin{Bmatrix} L_i \\ L_j \\ L_k \end{Bmatrix} = \frac{1}{2A} \begin{Bmatrix} a_i + b_{ir} + c_i z \\ a_j + b_{jr} + c_j z \\ a_k + b_{kr} + c_k z \end{Bmatrix} \rightarrow 17b$$

$$A = \frac{1}{2}\left(c_i z_j + r_j z_k + r_k z_i - r_j z_k - r_j z_i - r_k z_j \right) \rightarrow 17c$$

❖ In the definition of J earlier, we have used the notation. $r_{ij} = r_i - r_j$ and $z_{ij} = z_i - z_j$

❖ The determinant of J is,

$$det\, J = r_{13} z_{23} - r_{23} z_{13}$$

Finite Element Modeling – Triangular Element

- Recall that, $|\det J| = 2A_e$. That is, the absolute value of the determinant of J equals twice the area of the element. The inverse relations for Equations 15 and 16 are given by ,

$$\begin{Bmatrix} \dfrac{\partial u}{\partial r} \\ \dfrac{\partial u}{\partial z} \end{Bmatrix} = J^{-1} \begin{Bmatrix} \dfrac{\partial u}{\partial \xi} \\ \dfrac{\partial u}{\partial \eta} \end{Bmatrix} \quad \text{and} \quad \begin{Bmatrix} \dfrac{\partial \omega}{\partial r} \\ \dfrac{\partial \omega}{\partial z} \end{Bmatrix} = J^{-1} \begin{Bmatrix} \dfrac{\partial \omega}{\partial \xi} \\ \dfrac{\partial \omega}{\partial \eta} \end{Bmatrix} \qquad \rightarrow 18$$

Where

$$J^{-1} = \frac{1}{\det J} \begin{pmatrix} r_{13} & -z_{13} \\ -r_{23} & z_{23} \end{pmatrix} \qquad \rightarrow 19$$

Finite Element Modeling – Triangular Element

- Introducing these transformation relationships into the strain- displacement relations, in equation 6 and using equation 13, we get,

$$\epsilon = \begin{Bmatrix} \dfrac{z_{23}(q_1 - q_5) - z_{13}(q_3 - q_5)}{\det J} \\ \dfrac{r_{23}(q_2 - q_6) - r_{13}(q_4 - q_6)}{\det J} \\ \dfrac{-r_{23}(q_1 - q_5) + r_{13}(q_3 - q_6) + z_{23}(q_2 - q_6) - z_{13}(q_4 - q_6)}{\det J} \\ \dfrac{N_1 q_1 + N_2 q_3 + N_3 q_5}{r} \end{Bmatrix} \rightarrow 20$$

- This equation can be written in matrix form as,

$$\boldsymbol{\epsilon} = \boldsymbol{Bq} \qquad \rightarrow 21$$

Finite Element Modeling – Triangular Element

❖ where the element strain-displacement matrix, of dimension (4 x 6), is given by,

$$B = \begin{bmatrix} \frac{z_{23}}{\det J} & 0 & \frac{z_{31}}{\det J} & 0 & \frac{z_{12}}{\det J} & 0 \\ 0 & \frac{z_{32}}{\det J} & 0 & \frac{z_{13}}{\det J} & 0 & \frac{z_{21}}{\det J} \\ \frac{z_{32}}{\det J} & \frac{z_{23}}{\det J} & \frac{z_{13}}{\det J} & \frac{z_{31}}{\det J} & \frac{z_{21}}{\det J} & \frac{z_{12}}{\det J} \\ \frac{N_1}{r} & 0 & \frac{N_2}{r} & 0 & \frac{N_3}{r} & 0 \end{bmatrix} \quad \rightarrow 22$$

❖ The element stiffness matrix ***[K*** $^{(e)}$***]*** is given by,

$$k^e = 2\pi \bar{r} A_e \bar{B}^T D \bar{B} \quad \rightarrow 23$$

Where,

$$\bar{r} = \frac{r_1 + r_2 + r_3}{3}$$

Finite Element Modeling – Triangular Element

- $2\pi\bar{r}A_e$ is the volume of the ring-shaped element shown in Fig.7. Also A_e , is given as ,

$$A_e = \frac{1}{2}\left|\det J\right| \qquad \rightarrow 24$$

- The body force term is given as,

$$2\pi\int_e u^T f r dA = q^T f^e \qquad \rightarrow 25$$

where the element body force vector f^e is given as,

$$f^e = \frac{2\pi\bar{r}A_e}{3}\left[\bar{f}_r, \bar{f}_z, \bar{f}_r, \bar{f}_z, \bar{f}_r, \bar{f}_z\right]^T \qquad \rightarrow 26$$

- The bar on f terms indicate that they are evaluated at the centroid.

Solved Problems

In fig. 8 a long cylinder of inside diameter 80mm and outside diameter 120mm snugly fits in a hole over its full length. The cylinder is then subjected to an internal pressure of 2 Mpa. Using two elements on the 10mm length shown, find the displacements at the inner radius.

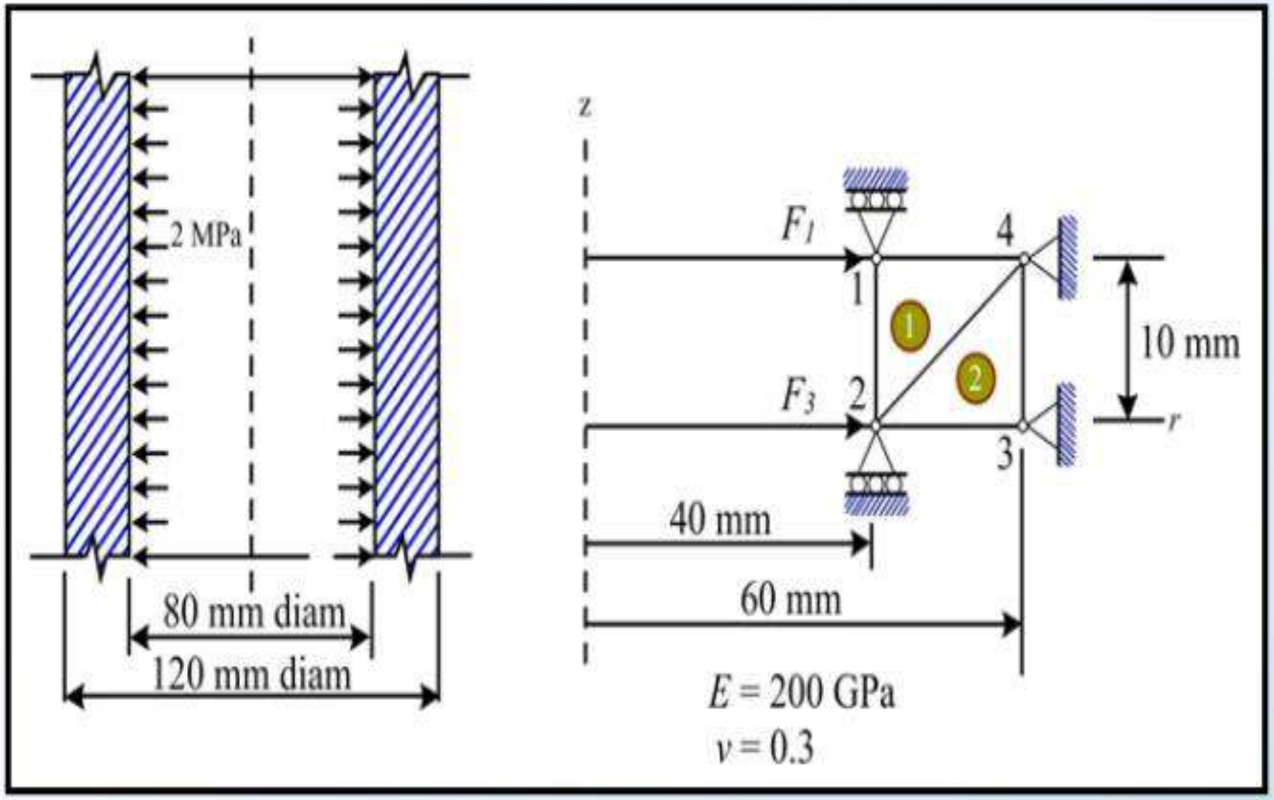


Figure 8

Solved Problems

Given Data :-

Inside Diameter = $80\ mm$.

Outside Diameter = $120\ mm$.

Internal Pressure = $2\ MPa$.

Young's Modulus (E) = $200 \times 10^9\ Pa$.

Poisson's Ratio (ν) = 0.3.

To Find :-

The displacements at the inner radius (Q_1 and Q_3). (Since roller supports are provided at the nodes 1 and 2. There is displacement only in x-direction & the displacement in y – direction is zero).

Solved Problems

Solution :-

Consider the following table:

Element	Connectivity		
	1	2	3
1	1	2	4
2	2	3	4

Node	Coordinates	
	r	z
1	40	10
2	40	0
3	60	0
4	60	10

We will use the units of millimeters for length, Newton's for force, and Mega-Pascal's for stress and *E*. These units are consistent. On substituting $E = 200000\ MPa$ and $\nu = 0.3$, we have,

$$D = \begin{bmatrix} 2.69\times10^5 & 1.15\times10^5 & 0 & 1.15\times10^5 \\ 1.15\times10^5 & 2.69\times10^5 & 0 & 1.15\times10^5 \\ 0 & 0 & 0.77 & 0 \\ 1.15\times10^5 & 1.15\times10^5 & 0 & 2.69\times10^5 \end{bmatrix}$$

Solved Problems

For both elements, *det J = 200 mm²* and A_e = *100 mm*. From Eq.24 , forces F_1 and F_3 are given by,

$$F_1 = F_3 = \frac{2\pi r_1 l_e P_i}{2} = \frac{2\pi(40)(10)(2)}{2} = 2514N$$

The B matrices relating element strains to nodal displacements are obtained first. For Element.1,

$$\bar{r} = \frac{1}{3}(40+40+60) = 46.67mm$$ and ,

$$\bar{B}^1 = \begin{bmatrix} -0.05 & 0 & 0 & 0 & 0.05 & 0 \\ 0 & 0.1 & 0 & -0.1 & 0 & 0 \\ 0.1 & -0.05 & -0.1 & 0 & 0 & 0.05 \\ 0.0071 & 0 & 0.0071 & 0 & 0.0071 & 0 \end{bmatrix}$$

Solved Problems

For element 2, $\bar{r} = \frac{1}{3}(40+60+60) = 53.33mm$ and ,

$$\overline{B}^2 = \begin{bmatrix} -0.05 & 0 & 0 & 0 & 0.05 & 0 \\ 0 & 0.1 & 0 & -0.1 & 0 & 0 \\ 0.1 & -0.05 & -0.1 & 0 & 0 & 0.05 \\ 0.00625 & 0 & 0.00625 & 0 & 0.00625 & 0 \end{bmatrix}$$

The element stress-displacement matrices are obtained by multiplying DB:

$$D\overline{B}^1 = 10^4 \begin{bmatrix} -1.26 & 1.15 & 0.082 & -1.15 & 1.43 & 0 \\ -0.49 & 2.69 & 0.082 & -2.69 & 0.657 & 0.1 \\ 0.77 & -0.385 & -0.77 & 0 & 0 & 0.385 \\ -0.384 & 1.15 & 0.191 & -1.15 & 0.766 & 0 \end{bmatrix}$$

$$D\overline{B}^2 = 10^4 \begin{bmatrix} -1.27 & 0 & 1.42 & -1.15 & 0.072 & 1.15 \\ -0.503 & 0 & 0.647 & -2.69 & 0.072 & 2.69 \\ 0 & -0.385 & -0.77 & 0.385 & 0.77 & 0 \\ -0.407 & 0 & 0.743 & -1.15 & 0.168 & 1.15 \end{bmatrix}$$

Solved Problems

The stiffness matrices are obtained by finding $2\pi \bar{r} A_e \bar{B}^T D \bar{B}$ for each element,

$$
\begin{array}{ll}
\text{Global d of} \rightarrow & \begin{array}{cccccc} 1 & 2 & 3 & 4 & 7 & 8 \end{array} \\
k^1 = 10^7 & \begin{bmatrix} 4.03 & -2.58 & -2.34 & 1.45 & -1.932 & 1.13 \\ & 8.45 & 1.37 & -7.89 & 1.93 & -0.565 \\ & & 2.30 & -0.24 & 0.16 & -1.13 \\ & & & 7.89 & -1.93 & 0 \\ & & & & 2.25 & 0 \\ \text{symmetric} & & & & & 0.565 \end{bmatrix}
\end{array}
$$

$$
\begin{array}{ll}
\text{Global d of} \rightarrow & \begin{array}{cccccc} 3 & 4 & 5 & 6 & 7 & 8 \end{array} \\
k^2 = 10^7 & \begin{bmatrix} 2.05 & 0 & -2.22 & 1.69 & -0.085 & -1.69 \\ & 0.645 & 1.29 & -0.645 & -1.29 & 0 \\ & & 5.11 & -3.46 & -2.42 & 2.17 \\ & & & 9.66 & 1.05 & -9.01 \\ & & & & 2.62 & 0.241 \\ \text{symmetric} & & & & & 9.01 \end{bmatrix}
\end{array}
$$

Using the elimination approach, on assembling the matrices with reference to the degrees of freedom 1 and 3, we get,

$$10^7 = \begin{bmatrix} 4.03 & -2.34 \\ -2.34 & 4.35 \end{bmatrix} \begin{Bmatrix} Q_1 \\ Q_1 \end{Bmatrix} = \begin{Bmatrix} 2514 \\ 2514 \end{Bmatrix}$$

Solved Problems

On performing matrix multiplication and solving the linear equations we obtain the values of displacements as,

$$Q_1 = 0.014 \times 10^{-2} mm$$

$$Q_2 = 0$$

$$Q_3 = 0.0133 \times 10^{-2} mm$$

$$Q_4 = 0$$

Result :-

The values of displacements at the nodes 1 and 2 on the inner surface of cylinder are,

$$Q_1 = 0.014 \times 10^{-2} mm$$

$$Q_2 = 0$$

$$Q_3 = 0.0133 \times 10^{-2} mm$$

$$Q_4 = 0$$

Solved Problems

2. A triangular axisymmetric ring element with nodes i, j, and k is shown in figure 9. The (r, z) coordinates of the nodes in centimeters are also indicated in figure 9. Find the shape functions corresponding to the nodal corresponding to the nodal degrees of freedom of the element.

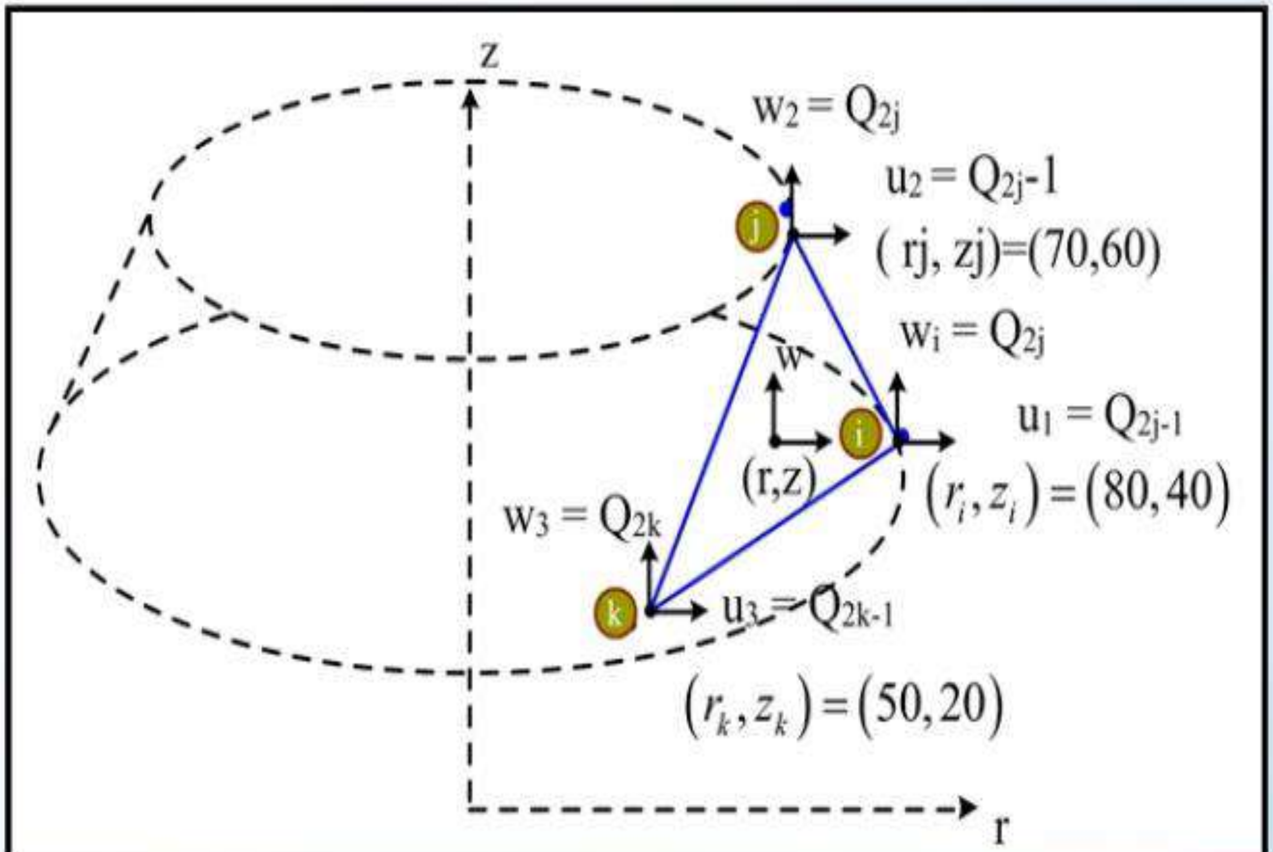


Figure 9

Solved Problems

Given Data:-

The co-ordinates at nodes i ,j , k.

$$(r_i, z_i) = (80, 40)$$

$$(r_j, z_j) = (70, 60)$$

$$(r_k, z_k) = (50, 20)$$

To find :-

The shape functions at the nodes

$$N_i(r, z)$$

$$N_j(r, z)$$

$$N_j(r, z)$$

Solved Problems

Solution :

The matrix of shape functions corresponding to the six degrees of freedom of the element is given by Eq. (11) with the shape functions defined in Eq.17 a and b. The constants $\mathbf{a}_i$,$\mathbf{a}_j$,$\mathbf{a}_k$ of can be found by replacing x and y by rand z, respectively, as,

$$a_i = r_j z_k - r_k z_j = 70(20) - 50(60) = -1600$$

$$a_j = r_k z_i - r_i z_k = 50(40) - 80(20) = 400$$

$$a_k = r_i z_j - r_j z_i = 80(60) - 70(40) = 2000$$

$$b_i = z_j - z_k = 60 - 20 = 40$$

$$b_k = z_i - z_j = 40 - 60 = -20$$

$$c_i = r_k - r_j = 50 - 70 = -20$$

$$c_j = r_i - r_k = 80 - 50 = 30$$

$$c_k = r_j - r_i = 70 - 80 = -10$$

Solved Problems

The area of triangle i, j, k is given by ,

$$A = \frac{1}{2}\left(r_i z_j + r_j z_k + r_k z_i - r_i z_k - r_j z_i - r_k z_i\right)$$

$$= \frac{1}{2}\left(80(60) + 70(60) + 50(40) - 80(20) - 70(40) - 50(60)\right)$$

$$= 400cm^2 = 0.04m^2$$

Thus the shape functions are defined as,

$$N_j(r,z) = \frac{1}{2A}\left(a_j + b_j r + c_j z\right) = 0.5 + 0.025r - 0.0375z$$

$$N_k(r,z) = \frac{1}{2A}\left(a_k + b_k r + c_k z\right) = 2.5 + 0.025r - 0.0125z$$

Solved Problems

For the above problem find matrix $[\underline{B}]$

Solution :-

The B matrix is given by the relation,

$$[B]=\frac{1}{2}A\begin{bmatrix} b_i & 0 & b_j & 0 & b_k & 0 \\ \left(\frac{N_i}{r}\right) & 0 & \left(\frac{N_j}{r}\right) & 0 & \left(\frac{N_k}{r}\right) & 0 \\ 0 & c_i & 0 & c_j & 0 & c_k \\ c_i & b_i & c_j & b_j & c_k & b_k \end{bmatrix}$$

On substituting the known values we obtain ,

$$[B]=\begin{bmatrix} 40 & 0 & -20 & 0 & -20 & 0 \\ \frac{1}{r}(-2+0.05_r-0.025_z) & 0 & \frac{1}{r}(0.5-0.025_r-0.037_z) & 0 & \frac{1}{r}(2.5+0.025_r-0.0125_z) & 0 \\ 0 & -20 & 0 & 30 & 0 & -10 \\ -20 & 40 & 30 & -20 & -10 & -20 \end{bmatrix}$$

Solved Problems

The centroid of the element is given by,

$$\{\underline{r_i}, \underline{z}\} = \left\{\frac{1}{3}(r_i + r_j + r_k), \frac{1}{3}(z_i + z_j + z_k)\right\} = \left\{\frac{1}{3}(80 + 70 + 50), \frac{1}{3}(40 + 60 + 20)\right\} = \{66.6667, 40.0\}\, cm$$

The matrix $[\underline{B}]$ can be determined by evaluating *[B] at the centroid, (r, z) = (r, z).*
Noting that,

$$\frac{1}{\underline{r}}(-2.0 + 0.05\underline{r} - 0.025\underline{z}) = \frac{1}{66.6667}\{-2.0 + 0.05(66.6667) - 0.025(40)\} = 0.005$$

$$\frac{1}{\underline{r}}(0.5 + 0.025\underline{r} - 0.0375\underline{z}) = \frac{1}{66.6667}\{0.5 + 0.025(66.6667) - 0.0375(40)\} = 0.005$$

$$\frac{1}{r}(2.5 - 0.025\,\mathrm{r} - 0.0125z) = \frac{1}{66.6667}\{2.5 - 0.025(66.6667) - 0.0125(40)\} = 0.005$$

Solved Problems

The matrix $\left[\underline{B}\right]$ can be found as,

$$\left[\underline{B}\right]=\left[B\left(\underline{r},\underline{z}\right)\right]=\begin{bmatrix} 0.05 & 0 & -0.025 & 0 & -0.025 & 0 \\ 0.005 & 0 & 0.005 & 0 & 0.005 & 0 \\ 0 & -0.025 & 0 & 0.0375 & 0 & -0.0125 \\ -0.025 & 0.05 & 0.0375 & -0.025 & -0.0125 & -0.025 \end{bmatrix}$$

Result :-

Value of $\left[\underline{B}\right]$ matrix is found out as,

$$\left[\underline{B}\right]=\left[B\left(\underline{r},\underline{z}\right)\right]=\begin{bmatrix} 0.05 & 0 & -0.025 & 0 & -0.025 & 0 \\ 0.005 & 0 & 0.005 & 0 & 0.005 & 0 \\ 0 & -0.025 & 0 & 0.0375 & 0 & -0.0125 \\ -0.025 & 0.05 & 0.0375 & -0.025 & -0.0125 & -0.025 \end{bmatrix}$$

Solved Problems

3. If the Young's modulus and the Poisson's ratio of the material of the triangular ring element ijk considered in problem 2 are given by 207 GPa and 0.3, respectively, Find the stiffness matrix of the element ?

Given Data :-

Young's modulus (E) = 207 Gpa.

Poisson's Ratio (v) = 0.3.

To find :-

The stiffness matrix of the element $[\mathbf{K}^{(e)}]$.

Solved Problems

Solution :

The stiffness matrix ***[K $^{(e)}$]*** is given by the equation 23 as,

$$k^e = 2\pi \bar{r} A_e \bar{B}^T D \bar{B}$$

we already know the values of $\left[\underline{B}\right]$. The value of D is calculated below,

$$[D] = \frac{207(10^9)}{(1.3)(0.4)} \begin{bmatrix} 0.7 & 0.3 & 0.3 & 0 \\ 0.3 & 0.7 & 0.3 & 0 \\ 0.3 & 0.3 & 0.7 & 0 \\ 0 & 0 & 0 & 0.2 \end{bmatrix} = 10^{11} \begin{bmatrix} 2.7865 & 1.1942 & 1.1942 & 0 \\ 1.1942 & 2.7865 & 1.1942 & 0 \\ 1.1942 & 1.1942 & 207865 & 0 \\ 0 & 0 & 0 & 0.7962 \end{bmatrix}$$

Noting that,

$$2\pi r A = 2\pi(66.6667)(400) = 1675.5208(10^2)cm^3 = 0.167552m^3$$

Solved Problems

The stiffness matrix of the triangular ring element ijk can be found out as,

$$\left[K^{(e)}\right]=10^{8}\begin{bmatrix} 1.3623 & -0.4419 & -0.6720 & -0.4961 & -0.5052 & -0.0542 \\ -0.4419 & 0.6253 & 0.3502 & -0.6045 & 0.0167 & -0.0208 \\ -0.6720 & 0.3502 & 0.4410 & -0.2751 & 0.1909 & -0.0750 \\ 0.4961 & -0.6045 & -0.2751 & 0.7399 & -0.1084 & -0.1355 \\ -0.5052 & 0.0167 & 0.1909 & -0.1084 & 0.2743 & 0.0917 \\ -0.0542 & -0.0208 & -0.0750 & -0.1355 & 0.0917 & 0.1563 \end{bmatrix}\frac{N}{m}$$

Result :-

The stiffness matrix of the given triangular axisymmetric element is calculated as,

$$\left[K^{(e)}\right]=10^{8}\begin{bmatrix} 1.3623 & -0.4419 & -0.6720 & -0.4961 & -0.5052 & -0.0542 \\ -0.4419 & 0.6253 & 0.3502 & -0.6045 & 0.0167 & -0.0208 \\ -0.6720 & 0.3502 & 0.4410 & -0.2751 & 0.1909 & -0.0750 \\ 0.4961 & -0.6045 & -0.2751 & 0.7399 & -0.1084 & -0.1355 \\ -0.5052 & 0.0167 & 0.1909 & -0.1084 & 0.2743 & 0.0917 \\ -0.0542 & -0.0208 & -0.0750 & -0.1355 & 0.0917 & 0.1563 \end{bmatrix}\frac{N}{m}$$

Stress Calculations

- From the set of nodal displacements obtained in the above problem, the element nodal displacements q can be found using the connectivity.
- Then, using stress-strain relation in Eq. 8 and strain-displacement relation in Eq.21, we have,

$$\sigma = D\overline{B}q \qquad \rightarrow 27$$

where, $[\underline{B}]$ is B evaluated at the centroid.

Higher Order
and
Isoparametric Elements

Learning Objectives

At the end of this topic, you will be able to:

- Describe about Higher Order One-dimensional Elements
- Explain about Higher Order Cubic Element
- Describe about Higher Order Elements in Terms of Natural Coordinates
- Illustrate the Problems on Higher Order and Isoparametric Elements
- Describe about Two-Dimensional Isoparametric Elements
- Explain about the Numerical Integration

Outcomes

By the end of this topic, you will be able to:

- Discuss about Higher Order One-dimensional Elements
- Describe Higher Order Cubic Element
- Understand Higher Order Elements In Terms Of Natural Coordinates
- Solve the Problems on Higher Order and Isoparametric Elements
- Explain Two-Dimensional Isoparametric Elements
- Discuss about the Numerical Integration

Higher Order and Isoparametric Elements

- If the interpolation polynomial is in the order of two or more, the element is known as a higher order element.
- A higher order element can be either complex or multiplex.
- In higher order elements, some secondary (mid-side and/or interior) nodes are introduced in addition to the primary (corner) nodes in order to match the number of nodal degrees of freedom with the number of constants (also known as generalized coordinates) in the interpolation polynomial.
- For problems involving curved boundaries, a family of elements known as isoparametric elements can be used.
- In isoparametric elements, the same interpolation functions used to define the element geometry are also used to describe the variation of the field variable within the element.

1. In a plane strain problem, we have

$$\sigma_x = 20\,000 \text{ psi}, \sigma_y = -10\,000 \text{ psi}$$
$$E = 30 \times 10^6 \text{ psi}, \nu = 0.3$$

Determine the value of the stress σ_z.

A)

Plane strain condition implies that

$$\varepsilon_z = 0 = -\nu\frac{\sigma_x}{E} - \nu\frac{\sigma_y}{E} + \frac{\sigma_z}{E}$$

which gives

$$\sigma_z = \nu(\sigma_x + \sigma_y)$$

We have, $\sigma_x = 20000$ psi $\sigma_y = -10000$ psi $E = 30 \times 10^6$ psi $\nu = 0.3$.

On substituting the values,

$$\sigma_z = 3000 \text{ psi}$$

2. If a displacement field is described by

$$u = (-x^2 + 2y^2 + 6xy)10^{-4}$$
$$v = (3x + 6y - y^2)10^{-4}$$

determine $\epsilon_x, \epsilon_y, \gamma_{xy}$ at the point $x = 1, y = 0$.

A)

Displacement field

$$u = 10^{-4}(-x^2 + 2y^2 + 6xy)$$
$$v = 10^{-4}(3x + 6y - y^2)$$
$$\frac{\partial u}{\partial x} = 10^{-4}(-2x + 6y) \quad \frac{\partial u}{\partial y} = 10^{-4}(4y + 6x)$$
$$\frac{\partial v}{\partial x} = 3 \times 10^{-4} \quad \frac{\partial v}{\partial y} = 10^{-4}(6 + 2y)$$
$$\varepsilon = \begin{Bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \end{Bmatrix}$$

at $x = 1, y = 0$

$$\varepsilon = 10^{-4}\begin{Bmatrix} -2 \\ 6 \\ 9 \end{Bmatrix}$$

3. **Consider the differential equation for a problem as**

$$\frac{d^2 y}{dx^2} + 300\, x^2 = 0, \qquad 0 \le x \le 1.$$

with the boundary conditions $y(0) = 0, y(1) = 0$. Find the solution of the problem using a one coefficient trial function as $y = a_1 x (1 - x^3)$. Use (i) Point collocation method, (ii) Sub-domain collocation method, (iii) Least square method and (iv) Galerkin's method.

A)

The given differential equation is

$$\frac{d^2 y}{dx^2} + 300\, x^2 = 0, \quad 0 \le x \le 1 \qquad \text{.... (1)}$$

The boundary conditions are $y = 0$ at $x = 0$ and $x = 1$

The trial function is $y = a_1 x (1 - x^3)$ .... (2)

First let us check this trial function whether it will satisfy the boundary conditions or not.

For $x = 0$, $y = a_1 (0)(1 - 0) = 0$

and for $x = 1$, $y = a_1 (1)(1 - 1^3) = 0$

Hence the selected trial function satisfies the boundary conditions.

Now the eqn. (2) $\Rightarrow y = a_1 (x - x^4)$

$$\frac{dy}{dx} = a_1 \left(1 - 4x^3\right) \text{ and } \frac{d^2 y}{dx^2} = a_1 (0 - 12x^2) = -12\, a_1 x^2$$

Substituting in the equation (1), we get the residual as $R = -12\, a_1 x^2 + 300\, x^2$

(i) Point Collocation Method:

In point collocation method, the residual is set to zero

i.e., $R = -12\, a_1 x^2 + 300\, x^2 = 0$

Since the number of coefficients to be found is one, (i.e., a_1), we require only one collocation point and this point should lie between 0 and 1.

Let us take $x = \frac{1}{2}$

Then $R = -12\, a_1 \left(\frac{1}{2}\right)^2 + 300 \left(\frac{1}{2}\right)^2 = 0$

i.e., $3a_1 = 75$ (or) $a_1 = 25$

Hence the trial function is $\mathbf{y = 25x (1 - x^3)}$ **(Answer)**

(ii) Sub-domain collocation Method:

In this method, the integral of the residual over the sub-domain is set to zero.

$$\text{i.e., } \int_0^1 R\,dx = 0$$

$$\text{i.e., } \int_0^1 \left(-12\,a_1\,x^2 + 300\,x^2 \right) dx = 0$$

$$\text{i.e., } \left[-12\,a_1 \frac{x^3}{3} + 300 \frac{x^3}{3} \right]_0^1 = 0$$

$$\text{i.e., } -4a_1 + 100 = 0$$

$$\text{(or) } a_1 = \frac{100}{4} = 25$$

Hence the trial function is $\mathbf{y = 25x\,(1 - x^3)}$ **(Answer)**

(iii) Least Square Method:

In this method, the integral of the square of the residual over the domain called as functional is to be minimum.

$$\text{i.e., The functional } I = \int_0^1 R^2\,dx = \text{minimum}$$

$$\text{Now } I = \int_0^1 R^2\,dx = \int_0^1 (-12\,a_1\,x^2 + 300\,x^2)^2\,dx$$

$$= \int_0^1 (144\,a_1^2\,x^4 + 90000\,x^4 - 7200\,a_1\,x^4)\,dx$$

$$= \left[144\,a_1^2 \frac{x^5}{5} + 90000 \frac{x^5}{5} - 7200\,a_1 \frac{x^5}{5} \right]_0^1$$

$$= \frac{144}{5}\,a_1^2 + \frac{90000}{5} - \frac{7200\,a_1}{5}$$

$$\text{Now } \frac{\partial I}{\partial a_1} = 0 \Rightarrow \frac{288}{5}\,a_1 - \frac{7200}{5} = 0$$

$$\text{(or) } 288\,a_1 = 7200$$

$$\text{i.e., } a_1 = \frac{7200}{288} = 25$$

Hence the trial function is $\mathbf{y = 25x\,(1 - x^3)}$ **(Answer)**

(iv) Galerkin's Method:

In this method, the domain integral of the product of the trial function with the residual is set to zero,

$$\text{i.e., } \int_0^1 (y \cdot R)\, dx = 0$$

$$\text{i.e., } \int_0^1 a_1 x (1 - x^3) \left(-12a_1 x^2 + 300 x^2 \right) dx = 0$$

$$\text{i.e., } \int_0^1 \left(a_1 x - a_1 x^4 \right)\left(-12 a_1 x^2 + 300 x^2 \right) dx = 0$$

$$\text{i.e., } \int_0^1 \left(-12 a_1^2 x^3 + 300 a_1 x^3 + 12 a_1^2 x^6 - 300 a_1 x^6 \right) dx = 0$$

$$\text{i.e., } \left[-12 a_1^2 \frac{x^4}{4} + 300 a_1 \frac{x^4}{4} + 12 a_1^2 \frac{x^7}{7} - 300 a_1 \frac{x^7}{7} \right]_0^1 = 0$$

$$\text{i.e., } -\frac{12 a_1^2}{4} + \frac{300}{4} a_1 + \frac{12}{7} a_1^2 - \frac{300}{7} a_1 = 0$$

Dividing by $(-12 a_1)$ we get

$$\frac{a_1}{4} - \frac{25}{4} - \frac{a_1}{7} + \frac{25}{7} = 0$$

$$\text{i.e., } \frac{7a_1 - 4a_1}{28} = \frac{175 - 100}{28}$$

(or) $3a_1 = 75$ i.e., $a_1 = 25$

Hence the trial function is $\mathbf{y = 25x\,(1 - x^3)}$ **(Answer)**

4. **Find the deflection at the centre of a simply supported beam of span length l subjected to a concentrated load P at its mid-point as shown in figure Use Rayleigh-Ritz method.**

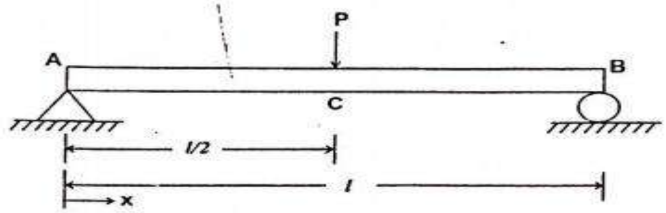


Fig.

As per Rayleigh-Ritz method, when the selected trial function containing Ritz coefficients makes the total potential energy minimum, that trial function is assumed as the approximate solution for the problem.

Now the problem is to find the deflection for a beam. Hence the total potential energy of beam is made to reach the minimum value by the trial function containing deflection term.

The total potential energy for a beam is given by

$$\pi = U - W$$

where U – Strain energy

W – Workdone by the external force

Strain energy for a beam,

$$U = \frac{EI}{2} \int_0^l \left(\frac{d^2 y}{dx^2} \right)^2 dx$$

where y is the deflection which can be expressed as

$$y = a_1 + a_2 x + a_3 x^2 + a_4 x^3 + a_5 x^4 + \ldots \qquad \ldots (1)$$

To simplify the problem, consider the first three terms such as

$$y = a_1 + a_2 x + a_3 x^2 \qquad \ldots (2)$$

The boundary conditions are $y = 0$ at $x = 0$ and $x = l$.

Hence eqn. (2) $\Rightarrow 0 = a_1$

and $0 = a_2 l + a_3 l^2$ which gives $a_2 = - a_3 l$

Then y can be expressed as

$$y = - a_3 l x + a_3 x^2 = a_3 (x^2 - l x)$$

Differentiating two times we get,

$$\frac{dy}{dx} = a_3 (2x - l) \text{ and } \frac{d^2 y}{dx^2} = 2a_3$$

Then, strain energy is given by

$$U = \frac{EI}{2} \int_0^l (2 a_3)^2 dx = \frac{EI}{2} 4a_3^2 l = 2EI a_3^2 l$$

Work done $W = P \cdot y_{max} = P \cdot y_{at\ x = l/2}$

$$= P\, a_3 \left(x^2 - l\, x \right)_{at\ x = l/2} \quad \text{(From Eqn.(3))}$$

$$= Pa_3 \left(\frac{l^2}{4} - \frac{l \cdot l}{2} \right)$$

$$= -P\, a_3 \frac{l^2}{4}$$

The total potential energy π is given by

$$\pi = U - W$$

$$= 2\,EI\, a_3^2\, l - \left(-Pa_3 \frac{l^2}{4} \right)$$

$$= 2\,EI\, a_3^2\, l + Pa_3 \frac{l^2}{4}$$

For minimum potential energy condition, $\frac{\partial \pi}{\partial a_3} = 0$

i.e., $4\,EI\, a_3\, l = -\frac{P\, l^2}{4}$

$$\therefore a_3 = -\frac{P\, l^2}{4} \times \frac{1}{4\,EI\, l} = -\frac{Pl}{16\,EI}$$

Substituting the value of a_3 in equation (3), we get

$$y = a_3\, (x^2 - l\, x) = -\frac{P\, l}{16\,EI}\, (x^2 - lx)$$

Maximum deflection occurs at $x = l/2$

$$\text{Hence } y_{max} = -\frac{P\, l}{16\,EI} \left(\frac{l^2}{4} - l \cdot \frac{l}{2} \right) = -\frac{P\, l}{16\,EI} \left(-\frac{l^2}{4} \right)$$

$$= \frac{P\, l^3}{64\,EI} \quad \text{which is the approximate solution}$$

But the exact solution is $y_{max} = \frac{P\, l^3}{48\,EI}$

To get more accurate solution by Rayleigh-Ritz method, the displacement function should contain more number of Ritz coefficients such as $y = a_1 + a_2\, x + a_3\, x^2 + a_4\, x^3 + a_5\, x^4$ and so on.

5. What are the Advantages and Disadvantages of FEM ?

Advantages of the finite element method:

1. The method can be used for any irregular-shaped domain and all types of boundary conditions.
2. Domains consisting of more than one material can be easily analyzed.
3. Accuracy of the solution can be improved either by proper refinement of the mesh or by choosing approximation of higher degree polynomials.
4. The algebraic equations can be easily generated and solved on a computer. In fact, a general purpose code can be developed for the analysis of a large class of problems.
5. This method can take care of any complex loading

Disadvantages of the finite element method:

1. The accuracy of results highly depends upon the degree of discretization (or meshing).
2. Manual judgement is essential in discretization process.
3. Finite element analysis requires large computer memory and time, and hence cost involved is high.
4. The method is complicated, and hence not viable for simple problems.
5. Mathematical background on the users part is required.

UNIT II

1. Explain the steps in FEM.

Step (i): Discretization of the structure

The first step in the finite element method is to divide the structure or solution region into subdivisions or elements. Hence, the structure is to be modeled with suitable finite elements. The number, type, size, and arrangement of the elements are to be decided.

Step (ii): Selection of a proper interpolation or displacement model

Since the displacement solution of a complex structure under any specified load conditions cannot be predicted exactly, we assume some suitable solution within an element to approximate the unknown solution. The assumed solution must be simple from a computational standpoint, but it should satisfy certain convergence requirements. In general, the solution or the interpolation model is taken in the form of a polynomial.

Step (iii): Derivation of element stiffness matrices and load vectors

From the assumed displacement model, the stiffness matrix $[K^e]$ and the load vector F^e of element e are to be derived by using either equilibrium conditions or a suitable variational principle.

Step (iv): Assemblage of element equations to obtain the overall equilibrium equations

Since the structure is composed of several finite elements, the individual element stiffness matrices and load vectors are to be assembled in a suitable manner and the overall equilibrium equations have to be formulated as

$$[K][Q] = F$$

where [K] is the assembled stiffness matrix, [Q] is the vector of nodal displacements, and F is the vector of nodal forces for the complete structure.

Step (v): Solution for the unknown nodal displacements

The overall equilibrium equations have to be modified to account for the boundary conditions of the problem. After the incorporation of the boundary conditions, the equilibrium equations can be expressed as

$$[K][Q] = F$$

For linear problems, the vector [Q] can be solved very easily. However, for nonlinear problems, the solution has to be obtained in a sequence of steps, with each step involving the modification of the stiffness matrix [K] and/or the load vector F.

Step (vi): Computation of element strains and stresses

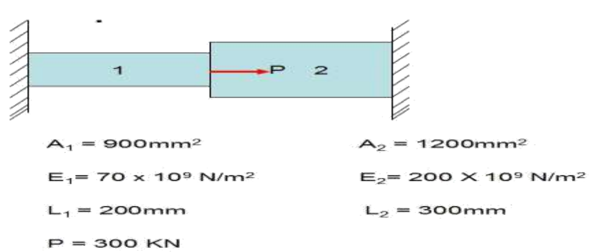
From the known nodal displacements [Q], if required, the element strains and stresses can be computed by using the necessary equations of solid or structural mechanics.

2. Consider the bar shown in Fig. An axial load $P = 300 \times 10^3$ N is applied as shown. Using the elimination approach for handling boundary conditions, do the following:

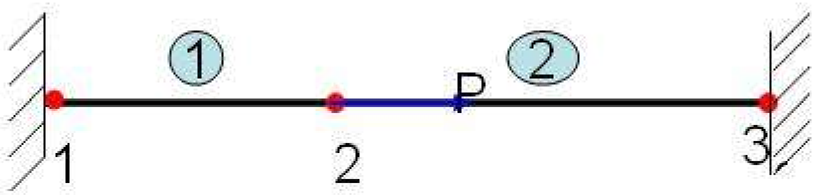
(a) Determine the nodal displacements

(b) Determine the stress in each element.

(c) Determine the reaction forces



To solve the system again the seven steps of FEM has to be followed, first 2 steps contain modeling and discretization. This result in



$$K_1 = \frac{A_1 E_1}{L_1}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \frac{900 \times 0.75 \times 10^5}{200}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = 10^5 \begin{array}{cc|c} 1 & 2 & \\ \multicolumn{2}{c}{\begin{bmatrix} 3.15 & -3.15 \\ -3.15 & 3.15 \end{bmatrix}} & \begin{array}{c} 1 \\ 2 \end{array} \end{array}$$

Similarly

$$K_2 = \frac{A_2 E_2}{L_2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = 10^5 \begin{array}{cc|c} 2 & 3 & \\ \multicolumn{2}{c}{\begin{bmatrix} 8 & -8 \\ -8 & 8 \end{bmatrix}} & \begin{array}{c} 2 \\ 3 \end{array} \end{array}$$

Next step is assembly which gives global stiffness matrix

$$K = \begin{array}{ccc|c} 1 & 2 & 3 & \\ \multicolumn{3}{c}{\begin{bmatrix} 3.15 & -3.15 & 0 \\ -3.15 & 3.15+8 & -8 \\ 0 & -8 & 8 \end{bmatrix} 10^5} & \begin{array}{c} 1 \\ 2 \\ 3 \end{array} \end{array}$$

Now determine global load vector

$$F = \begin{pmatrix} F_1 \\ F_2 \\ F_3 \end{pmatrix} = \begin{pmatrix} R_1 \\ 300 \times 10^3 \\ R_3 \end{pmatrix}$$

We have the equilibrium condition KQ=F

$$10^5\begin{bmatrix}3.15 & -3.15 & 0\\ -3.15 & 3.15+8 & -8\\ 0 & -8 & 8\end{bmatrix}\begin{Bmatrix}Q1\\ Q2\\ Q3\end{Bmatrix} = \begin{Bmatrix}R_1\\ 300\times 10^3\\ R_3\end{Bmatrix}\begin{matrix} \\ -(-3.15 \text{ X } 10^5 \text{ X} Q1)\\ -(0\text{X}Q1)\end{matrix}$$

After applying elimination method we have Q2 = 0.26mm

a) Nodal displacements = $[0, 0.26, 0]^T$ mm

Once displacements are known stress components are calculated as follows

b) Stress in each element:

For element 1

$$\sigma_1 = E_1 \frac{1}{L_1}[-1 \quad 1]\begin{Bmatrix}Q1\\ Q2\end{Bmatrix} = 94.17 \text{ N/mm}^2$$

For element 2

$$\sigma_2 = E_2 \frac{1}{L_2}[-1 \quad 1]\begin{Bmatrix}Q2\\ Q3\end{Bmatrix} = -179.34 \text{ N/mm}^2$$

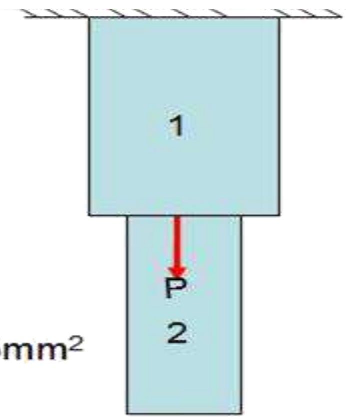
c) Supports Reaction:

$$\mathbf{R_1 = K_{11}Q_1 + K_{12}Q_2 + \ldots\ldots\ldots\ldots\ldots + K_{1N}Q_N - F_1}$$

$$\mathbf{R_1} = 10^5\,[3.15 \;\; -3.15 \;\; 0]\begin{bmatrix}0\\ 0.26\\ 0\end{bmatrix} = -0.819\text{X}10^5 \text{ N}$$

$$R_3 = 10^5\,[0 \;\; -8 \;\; 8]\begin{bmatrix}0\\ 0.26\\ 0\end{bmatrix} = -2.08 \text{ X}10^5 \text{ N}$$

3. Consider the bar shown in Fig. Using the elimination approach for handling boundary conditions. Determine the nodal displacements.



$E_1 = 2.06 \times 10^5$ MPa

$A_1 = 3387.09\text{mm}^2$ $A_2 = 2419.35\text{mm}^2$

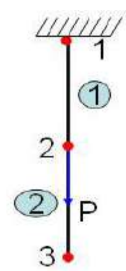
$L_1 = L_2 = 304.8\text{mm}$

$P = 444.8\text{N}$

Body force $= f_b = 7.69 \times 10^{-5}\text{N/mm}^3$

Solution:

$$K_1 = \frac{A_1E_1}{L_1}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = 10^6 \begin{array}{cc} 1 & 2 \\ \left[\begin{matrix} 2.28 & -2.28 \\ -2.28 & 2.28 \end{matrix}\right] & \begin{matrix} 1 \\ 2 \end{matrix} \end{array}$$

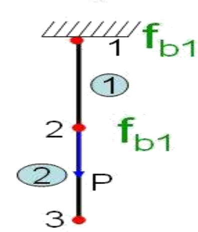


$$K_2 = \frac{A_2E_2}{L_2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = 10^6 \begin{array}{cc} 2 & 3 \\ \left[\begin{matrix} 1.63 & -1.63 \\ -1.63 & 1.63 \end{matrix}\right] & \begin{matrix} 2 \\ 3 \end{matrix} \end{array}$$

$$K = \begin{array}{cc} \begin{matrix} 1 & \quad 2 & \quad 3 \end{matrix} & \\ \left[\begin{matrix} 2.28 & -2.28 & 0 \\ -2.28 & 2.28+1.63 & -1.63 \\ 0 & -1.63 & 1.63 \end{matrix}\right] 10^6 & \begin{matrix} 1 \\ 2 \\ 3 \end{matrix} \end{array}$$

Body force terms

Element 1



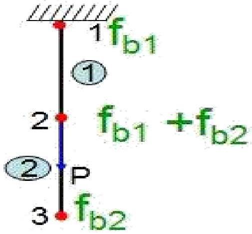
$$f_{b1} = \frac{A_1 f_b L_1}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix}\begin{matrix} 1 \\ 2 \end{matrix}$$

$$= \frac{3387.09 \times 7.69 \times 10^{-5} \times 304.8}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix}\begin{matrix} 1 \\ 2 \end{matrix}$$

$$= \begin{bmatrix} 39.69 \\ 39.69 \end{bmatrix}\begin{matrix} 1 \\ 2 \end{matrix}$$

Body force terms

Element 2

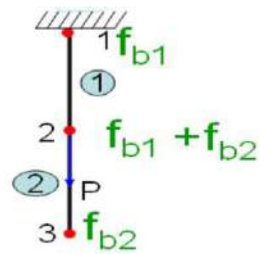


$$f_{b2} = \frac{A_2 f_b L_2}{2} \begin{array}{c} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} \end{array} \begin{array}{l} 2 \\ 3 \end{array}$$

$$= \frac{2419.35 \times 7.69 \times 10^{-5} \times 304.8}{2} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} \begin{array}{l} 2 \\ 3 \end{array}$$

$$= \begin{Bmatrix} 28.3 \\ 28.3 \end{Bmatrix} \begin{array}{l} 2 \\ 3 \end{array}$$

Global load vector:



$$F = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix} = \begin{Bmatrix} f_{b1} \\ p + f_{b1} + f_{b2} \\ fb2 \end{Bmatrix} = \begin{Bmatrix} 39.69 \\ 512.8 \\ 28.3 \end{Bmatrix}$$

$$10^6 \begin{bmatrix} 2.28 & -2.28 & 0 \\ -2.28 & 6.92 & -16.3 \\ 0 & -1.63 & 1.63 \end{bmatrix} \begin{Bmatrix} Q1 \\ Q2 \\ Q3 \end{Bmatrix} = \begin{Bmatrix} 39.69 + R1 \\ 512.8 \\ 28.3 \end{Bmatrix}$$

$Q2 = 0.23 \times 10^{-3}$mm

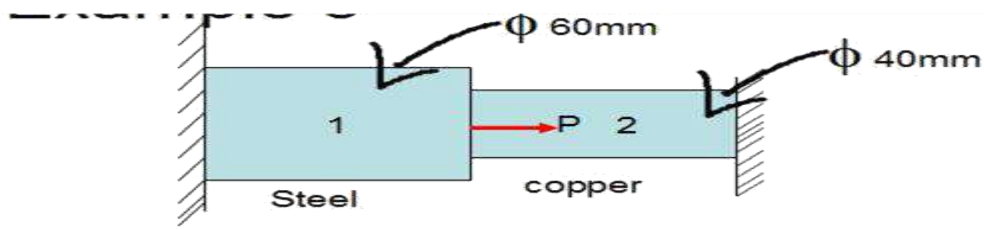
$Q3 = 2.5 \times 10^{-4}$mm

After applying elimination method and solving matrices we have the value of displacements as $Q2 = 0.23 \times 10^{-3}$mm & $Q3 = 2.5 \times 10^{-4}$mm

4. Consider the bar shown in Fig. An axial load $P = 100 \times 10^3$ N is applied as shown. Using the penalty approach for handling boundary conditions, do the following:

(a) Determine the nodal displacements

(b) Determine the reaction forces

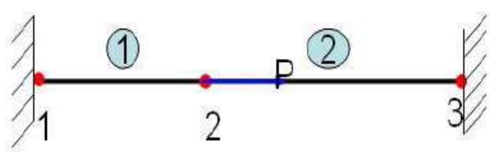


$E_1 = 2 \times 10^5$ MPa $E_2 = 1 \times 10^5$ MPa

$L_1 = 800$ mm $L_2 = 500$mm

$P = 100$ KN

Solution:



$$A_1 = \pi/4\,(60)^2 = 2827.43\text{mm}^2$$

$$A_2 = \pi/4\,(40)^2 = 1256.63\text{mm}^2$$

$$K_1 = \frac{A_1E_1}{L_1}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \frac{2827.43\text{x}2\text{x}\,10^5}{800}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = 10^5 \begin{array}{c} \begin{array}{cc} 1 & \quad 2 \end{array} \\ \begin{bmatrix} 7.06 & -7.06 \\ -7.06 & 7.06 \end{bmatrix} \end{array} \begin{array}{c} 1 \\ 2 \end{array}$$

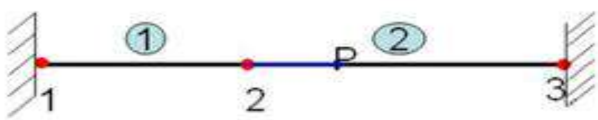
$$K_2 = \frac{A_2E_2}{L_2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = 10^5 \begin{array}{c} \begin{array}{cc} 2 & \quad 3 \end{array} \\ \begin{bmatrix} 2.51 & -2.51 \\ -2.51 & 2.51 \end{bmatrix} \end{array} \begin{array}{c} 2 \\ 3 \end{array}$$

Global stiffness matrix

$$K = \begin{array}{c} \\ \end{array}\begin{array}{ccc} 1 & 2 & 3 \end{array}$$

$$K = \begin{bmatrix} 7.07 & -7.07 & 0 \\ -7.07 & 9.583 & -2.513 \\ 0 & -2.513 & 2.513 \end{bmatrix} 10^5 \begin{array}{c} 1 \\ 2 \\ 3 \end{array}$$

Global load vector:



$$F = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 100 \times 10^3 \\ 0 \end{Bmatrix}$$

Equilibrium Equation

$$K\,Q = F$$

$$K = \begin{bmatrix} 7.07 & -7.07 & 0 \\ -7.07 & 9.583 & -2.513 \\ 0 & -2.513 & 2.513 \end{bmatrix} 10^5 \begin{Bmatrix} Q1 \\ Q2 \\ Q3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 100 \times 10^3 \\ 0 \end{Bmatrix}$$

$$C = \max [\, K_{ij} \,] \times 10^4 = 9.583 \times 10^5 \times 10^4$$

Modification required

$$\begin{pmatrix} 7.07 + C & -7.07 & 0 \\ -7.07 & 9.583 & -2.513 \\ 0 & -2.513 & 2.513 + C \end{pmatrix} 10^5 \begin{pmatrix} Q1 \\ Q2 \\ Q3 \end{pmatrix} = \begin{pmatrix} 0 + C\,a1 \\ 100 \times 10^3 \\ 0 + C\,a3 \end{pmatrix}$$

After Modification

$$\begin{pmatrix} 9.583 \times 10^4 & -7.07 & 0 \\ -7.07 & 9.583 & -2.513 \\ 0 & -2.513 & 9.583 \times 10^4 \end{pmatrix} 10^5 \begin{pmatrix} Q1 \\ Q2 \\ Q3 \end{pmatrix} = \begin{pmatrix} 0 \\ 100 \times 10^3 \\ 0 \end{pmatrix}$$

Solving the matrix we have

$Q1 = 7.698 \times 10^{-6}$mm, $Q2 = 0.104$mm, $Q3 = 2.736 \times 10^{-6}$mm

Reaction forces

@ node 1

$$R_1 = C(Q1 - a1) = -73597.44\text{N}$$

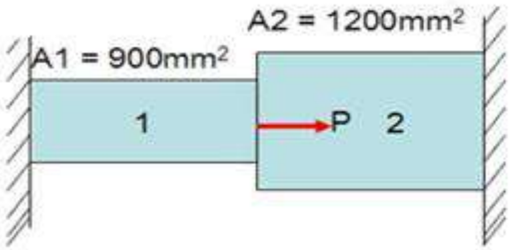
@ node 3

$$R_3 = C(Q3 - a3) = -26219.08\text{N}$$

5.An axial load P 300 KN is applied at 20^0C to the rod as shown m Fig.The temperature is then raised to 60^0C.

(a) Assemble the K and F matrices.

(b) Determine the nodal displacements and element stresses



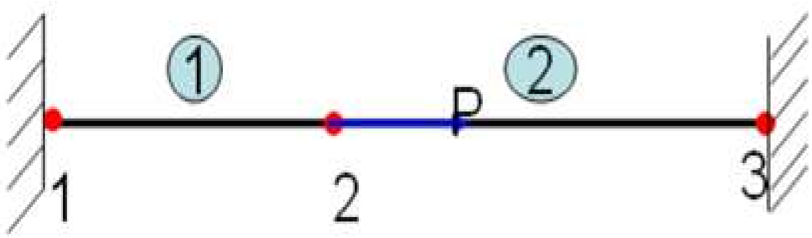
$\alpha_1 = 23 \times 10^{-6}$ Per ^{0}C $\qquad \alpha_2 = 11.7 \times 10^{-6}$ Per ^{0}C

$E_1 = 70 \times 10^9$ N/m^2 $\quad E_2 = 200 \times 10^9$ N/m^2

$L_1 = 200$mm $\qquad L_2 = 300$mm

P = 300 KN is applied at 20^0c ,the temperature is then raised to 60^0c

Solution:



$$K_1 = \frac{A_1E_1}{L_1}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \frac{900 \times 70 \times 10^3}{200}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = 10^3 \begin{array}{cc} 1 & 2 \\ \begin{bmatrix} 315 & -315 \\ -315 & 315 \end{bmatrix} & \begin{array}{c} 1 \\ 2 \end{array} \end{array}$$

$$K_2 = \frac{A_2E_2}{L_2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = 10^3 \begin{array}{cc} 2 & 3 \\ \begin{bmatrix} 800 & -800 \\ -800 & 800 \end{bmatrix} & \begin{array}{c} 2 \\ 3 \end{array} \end{array}$$

Global stiffness matrix:

$$K=\begin{array}{c c} \begin{array}{ccc} 1 & 2 & 3 \end{array} & \\ \begin{pmatrix} 315 & -315 & 0 \\ -315 & 1115 & -800 \\ 0 & -800 & 800 \end{pmatrix} 10^3 & \begin{array}{c} 1 \\ 2 \\ 3 \end{array} \end{array}$$

Thermal load vector:

We have the expression of thermal load vector given by

$$\theta = EA\alpha\Delta T\begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Element 1

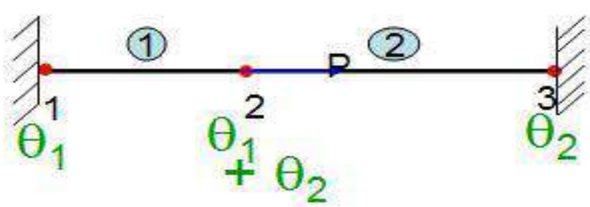
$$\theta_1 = 70 \times 10^3 \times 900 \times 23 \times 10^{-6} \times 40 \begin{bmatrix} -1 \\ 1 \end{bmatrix}\begin{array}{c} 1 \\ 2 \end{array}$$

$$\theta_1 = 10^3 \begin{pmatrix} -57.96 \\ 57.96 \end{pmatrix}\begin{array}{c} 1 \\ 2 \end{array}$$

Similarly calculate thermal load distribution for second element

$$\theta_2 = 10^3 \begin{pmatrix} -112.32 \\ 112.32 \end{pmatrix}\begin{array}{c} 2 \\ 3 \end{array}$$

Global load vector:



$$F=\begin{pmatrix} F_1 \\ F_2 \\ F_3 \end{pmatrix} = \begin{pmatrix} \theta_1 \\ P+\theta_2+\theta_1 \\ \theta_2 \end{pmatrix} = \begin{bmatrix} -57.96 \\ 245.64 \\ 112.32 \end{bmatrix} 10^3$$

From the equation KQ=F we have

$$\begin{bmatrix} 315 & -315 & 0 \\ -315 & 1115 & -800 \\ 0 & -800 & 800 \end{bmatrix} 10^3 \begin{bmatrix} Q1 \\ Q2 \\ Q3 \end{bmatrix} = \begin{bmatrix} -57.96 \\ 245.64 \\ 112.32 \end{bmatrix} \begin{matrix} \\ 10^3 \; -(-315 \times 10^3)Q1-(-8 \times 10^5)Q3 \\ -(0)Q1 \end{matrix}$$

After applying elimination method and solving the matrix we have Q_2= 0.22mm

Stress in each element:

For element 1

$$\sigma_1 = E_1 \frac{1}{L_1} [-1 \quad 1] \begin{bmatrix} Q1 \\ Q2 \end{bmatrix} - E_1 \alpha_1 \Delta T$$

$$= 12.60\text{MPa}$$

For element 2

$$\sigma_2 = E_2 \frac{1}{L_2} [-1 \quad 1] \begin{bmatrix} Q2 \\ Q3 \end{bmatrix} - E_2 \alpha_2 \Delta T$$

$$= -240.27\text{MPa}$$

6. Define discretization of domain and what are the steps in discretization procedure?

The process of dividing the domain into discrete elements is called discretization.

DISCRETIZATION PROCEDURES:

1. Type of Elements
2. Size of Elements
3. Location of Nodes
4. Number of Elements
5. Simplifications Afforded by the Physical Configuration of the Body
6. Finite Representation of Infinite Bodies

UNIT III

1. What are the assumptions of Trusses?

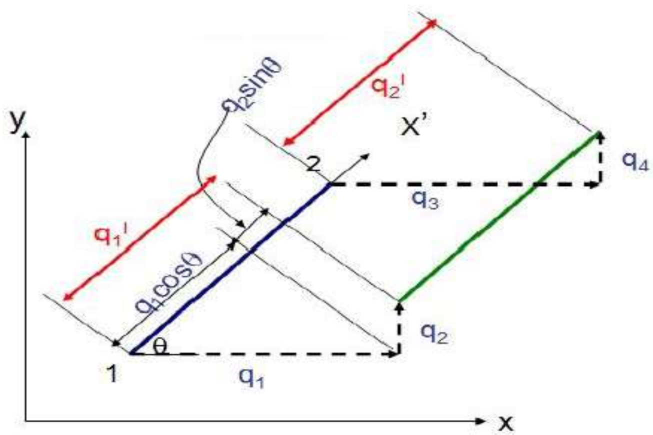
The following are the assumptions of Truss element

1. Truss element is only a prismatic member ie cross sectional area is uniform along its length.
2. It should be an isotropic material.
3. Constant load that is load is independent of time.
4. Homogenous material.
5. A load on a truss can only be applied at the joints (nodes).
6. Due to the load applied each bar of a truss is either induced with tensile/compressive forces.
7. The joints in a truss are assumed to be frictionless pin joints.
8. Self-weight of the bars are neglected.

2. Derive Transformation matrix (L) and element stiffness matrix for truss element and also Stress in each element.

Derivation of Transformation matrix (L):

Consider one truss element as shown that has nodes 1 and 2 .The coordinate system that passes along the element (x^1 axis) is called local coordinate and X-Y system is called as global coordinate system. After the loads applied let the element takes new position say locally node 1 has displaced by an amount q_1^1 and node2 has moved by an amount equal to q_2^1.As each node has 2 dof in global coordinate system. Let node 1 has displacements q_1 and q_2 along x and y axis respectively similarly q_3 and q_4 at node 2.



Resolving the components q_1,q_2, q_3 and q_4 along the bar we get two equations as

$q_1^1 \;= q_1 \cos\theta + \; q_2 \sin\theta$

$q_2^1 \;= q_3 \cos\theta + \; q_4 \sin\theta$

Or

$q_1^1 \;= q_1 l + \; q_2 m$

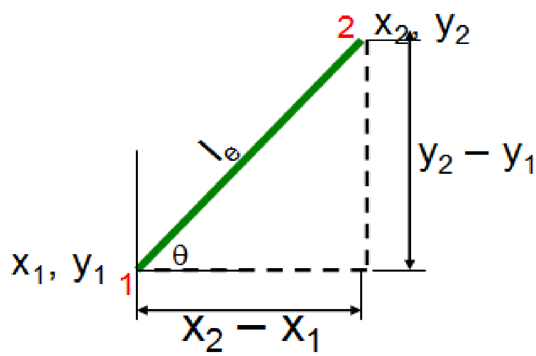
$q_2^1 \;= q_3 l + \; q_4 m$

Writing the same equation into the matrix form

$$\begin{bmatrix} q_1^1 \\ q_2^1 \end{bmatrix} = \begin{bmatrix} l & m & 0 & 0 \\ 0 & 0 & l & m \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix}$$

The above equation similar to q^1 = Lq
Where L is called transformation matrix that is used for local –global correspondence.
l,m are called direction cosines
<u>How to calculate direction cosines</u>

Consider a element that has node 1 and node 2 inclined by an angle θ as shown .let (x_1, y_1) be the coordinate of node 1 and (x_2,y_2) be the coordinates at node 2. When orientation of an element is know we use this angle to calculate l and m as:



$l = \cos\theta \quad m = \cos(90 - \theta) = \sin\theta$

and by using nodal coordinates we can calculate using the relation

$l = \frac{X_2 - X_1}{l_e} \qquad m = \frac{Y_2 - Y_1}{l_e}$

We can calculate length of the element as

$l_e = \sqrt{(X_2 - X_1)^2 + (Y_2 - Y_1)^2}$

<u>Derivation of element stiffness matrix for truss element</u>

Strain energy for a bar element we have U = ½ q^TKq

For a truss element we can write U = ½ q^{1T}Kq^1

Where q^1= L q and q^{1T}= $L^T q^T$

Therefore

U = ½ q^{1T}Kq^1

= ½$L^T q^T$K L q

= ½$q^T (L^T KL)$q

= ½$q^T K^e$q

Where K^e is the stiffness matrix of truss element

$K^e = L^T KL$

L = $\begin{bmatrix} l & m & 0 & 0 \\ 0 & 0 & l & m \end{bmatrix}$ $\quad L^T = \begin{bmatrix} l & 0 \\ m & 0 \\ 0 & l \\ 0 & m \end{bmatrix}$ $\quad$ K = $\frac{EA}{l_e}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$

Taking the product of all these matrix we have stiffness matrix for truss element which is given as

$$K^e = \begin{bmatrix} l & 0 \\ m & 0 \\ 0 & l \\ 0 & m \end{bmatrix} \frac{EA}{l_e} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} l & m & 0 & 0 \\ 0 & 0 & l & m \end{bmatrix} = \frac{EA}{l_e} \begin{bmatrix} l^2 & lm & -l^2 & -lm \\ lm & m^2 & -lm & -m^2 \\ -l^2 & -lm & l^2 & lm \\ -lm & -m^2 & lm & m^2 \end{bmatrix}$$

<u>Stress component for truss element</u>

The stress σ in a truss element is given by $\sigma = \epsilon E$ But strain ϵ= B ql and ql = T q

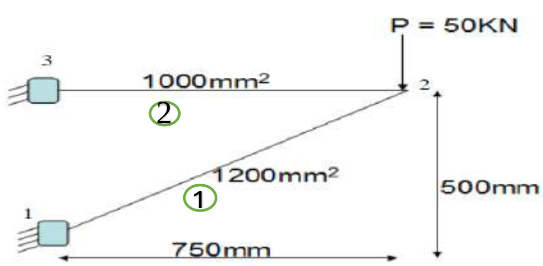
Where B = $\frac{1}{L}\begin{bmatrix} -1 & 1 \end{bmatrix}$

Therefore

$$\sigma = \frac{E}{L_e}\begin{bmatrix} -l & -m & l & m \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix}$$

3. For the two-bar truss shown in figure, determine the displacements. Stress in each element and support reactions. Take E=2x10^5 N/mm^2.

Solution: For given structure if node numbering is not given we have to number them which depend on user. Each node has 2 dof say q_1 q_2 be the displacement at node 1, q_3 & q_4 be displacement at node 2, q_5 &q_6 at node 3.



Nodal coordinate table

Node	x	y
1	0	0
2	750	500
3	0	500

Element connectivity table

element	Node 1	Node 2
1	1	2
2	2	3

Directional cosine table

element	l_e	l	m
1	901.3	0.832	0.554
2	750	-1	0

$$K^e = \frac{EA}{l_e}\begin{bmatrix} l^2 & lm & -l^2 & -lm \\ lm & m^2 & -lm & -m^2 \\ -l^2 & -lm & l^2 & lm \\ -lm & -m^2 & lm & m^2 \end{bmatrix}$$

$$K^1 = 10^5 \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{pmatrix} 1.84 & 1.22 & -1.84 & -1.22 \\ 1.22 & 0.816 & -1.22 & -0.816 \\ -1.84 & -1.22 & 1.84 & 1.22 \\ -1.22 & -0.816 & 1.22 & 0.816 \end{pmatrix} \end{array} \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} \qquad K^2 = 10^5 \begin{array}{c} \begin{array}{cccc} 3 & 4 & 5 & 6 \end{array} \\ \begin{pmatrix} 2.66 & 0 & -2.66 & 0 \\ 0 & 0 & 0 & 0 \\ -2.66 & 0 & 2.66 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{array} \begin{array}{c} 3 \\ 4 \\ 5 \\ 6 \end{array}$$

Element 1 has displacements q_1,q_2, q_3,q_4. Hence numbering scheme for the first stiffness matrix (K^1) as 1 2 3 4 similarly for K^2 3 4 5 & 6 as shown above.

Global stiffness matrix: the structure has 3 nodes at each node 3 dof hence size of global stiffness matrix will be 3 X 2 = 6 ie 6 X 6

$$K = 10^5 \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{pmatrix} 1.84 & 1.22 & -1.84 & -1.22 & 0 & 0 \\ 1.22 & 0.816 & -1.22 & -0.816 & 0 & 0 \\ -1.84 & -1.22 & 4.5 & 1.22 & -2.66 & 0 \\ -1.22 & -0.816 & 1.22 & 0.816 & 0 & 0 \\ 0 & 0 & -2.66 & 0 & 2.66 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array} \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array}$$

Global force vector F = $\begin{bmatrix} 0 \\ 0 \\ 0 \\ -50X10^3 \\ 0 \\ 0 \end{bmatrix}$

Equilibrium equation KQ = F

$$10^5 \begin{bmatrix} 1.84 & 1.22 & -1.84 & -1.22 & 0 & 0 \\ 1.22 & 0.816 & -1.22 & -0.816 & 0 & 0 \\ -1.84 & -1.22 & 4.5 & 1.22 & -2.66 & 0 \\ -1.22 & -0.816 & 1.22 & 0.816 & 0 & 0 \\ 0 & 0 & -2.66 & 0 & 2.66 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -50X10^3 \\ 0 \\ 0 \end{bmatrix}$$

Since node 1 is fixed q_1=q_2=0 and also at node 3 q_5 = q_6= 0 .At node 2 q_3& q_4 are free hence has displacements. In the load vector applied force is at node 2 ie F_4 = 50KN rest other forces zero.

By using elimination method, we can eliminate 1st row & column, 2nd row & column, 5th row & column and 6th row & column in equilibrium equation

The matrix reduced to 2X2 matrix

$$10^5 \begin{bmatrix} 4.5 & 1.22 \\ 1.22 & 0.816 \end{bmatrix} \begin{bmatrix} q_3 \\ q_4 \end{bmatrix} = \begin{bmatrix} 0 \\ -50X10^3 \end{bmatrix}$$

Solving above equation we get q_3 = 0.28mm, q_4 = -1.03mm

STRESS IN THE ELEMENT:

$$\sigma = \frac{E}{L_e}\begin{bmatrix}-l & -m & l & m\end{bmatrix}\begin{bmatrix}q_1\\q_2\\q_3\\q_4\end{bmatrix}$$

Stress in the 1st element

$$\sigma_1 = \frac{E}{L_e}\begin{bmatrix}-l & -m & l & m\end{bmatrix}\begin{bmatrix}q_1\\q_2\\q_3\\q_4\end{bmatrix}$$

$$\sigma_1 = \frac{2X10^5}{901.3}\begin{bmatrix}-0.832 & -0.554 & 0.832 & 0.554\end{bmatrix}\begin{bmatrix}0\\0\\0.28\\-1.03\end{bmatrix} = \text{-74.92 N/mm}^2$$

$$\sigma_2 = \frac{2X10^5}{750}\begin{bmatrix}1 & 0 & -1 & 0\end{bmatrix}\begin{bmatrix}0.28\\-1.03\\0\\0\end{bmatrix} = \text{-74.66 N/mm}^2$$

Support reactions:

$$\mathbf{R_1} = 10^5\begin{bmatrix}1.84 & 1.22 & -1.84 & -1.22 & 0 & 0\end{bmatrix}\begin{bmatrix}0\\0\\0.28\\-1.03\\0\\0\end{bmatrix} - [0] = 74.14 \text{ KN}$$

$$R_2 = 10^5\begin{bmatrix}1.22 & 0.816 & -1.22 & -0.816 & 0 & 0\end{bmatrix}\begin{bmatrix}0\\0\\0.28\\-1.03\\0\\0\end{bmatrix} - [0] = 49.88 \text{ KN}$$

$$\mathbf{R_5} = 10^5\begin{bmatrix}0 & 0 & -2.66 & 0 & 2.66 & 0\end{bmatrix}\begin{bmatrix}0\\0\\0.28\\-1.03\\0\\0\end{bmatrix} - [0] = -74.48 \text{ KN}$$

$$R_6 = 10^5[0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0]\begin{bmatrix} 0 \\ 0 \\ 0.28 \\ -1.03 \\ 0 \\ 0 \end{bmatrix} - [0\,] = 0$$

4. Derive Hermite shape functions for beam element.

1D linear beam element has two end nodes and at each node 2 dof which are denoted as Q_{2i-1} and Q_{2i} at node i. Here Q_{2i-1} represents transverse deflection where as Q_{2i} is slope or rotation. Consider a beam element has node 1 and 2 having dof as shown.

The shape functions of beam element are called as Hermite shape functions as they contain both nodal value and nodal slope which is satisfied by taking polynomial of cubic order

$$H_i = a_i + b_i\,\xi + c_i\,\xi^2 + d_i\,\xi^3$$

That must satisfy the following conditions

ξ	H_1	H_1'	H_2	H_2'	H_3	H_3'	H_4	H_4'
$\xi = -1$	1	0	0	1	0	0	0	0
$\xi = 1$	0	0	0	0	1	0	0	1

Applying these conditions determine values of constants as

$$H_i = a_i + b_i\,\xi + c_i\,\xi^2 + d_i\,\xi^3$$

@ node 1

$$H_1 = 1, \quad H_1' = 0, \quad \xi = -1$$

$$1 = a_1 - b_1 + c_1 - d_1 \longrightarrow (1)$$

$$H_1' = \frac{dH_1}{d\xi} = 0 = b_1 - 2c_1 + 3d_1 \longrightarrow (2)$$

$$H_i = a_i + b_i\,\xi + c_i\,\xi^2 + d_i\,\xi^3$$

@ node 2

$$H_1 = 1,\ \ H_1' = 0\ ,\ \ \xi = 1$$

$$0 = a_1 + b_1 + c_1 + d_1 \longrightarrow (3)$$

$$H_1' = \frac{dH_1}{d\xi} = 0 = b_1 + 2c_1 + 3d_1 \longrightarrow (4)$$

Solving above 4 equations we have the values of constants

$$a_1 = \tfrac{1}{2}\ ,\ b_1 = -\tfrac{3}{4}\ ,\ c_1 = 0,\ \ d_1 = \tfrac{1}{4}$$

Therefore

$$H_1 = \tfrac{1}{4}\,(2 - 3\xi + \xi^3)$$

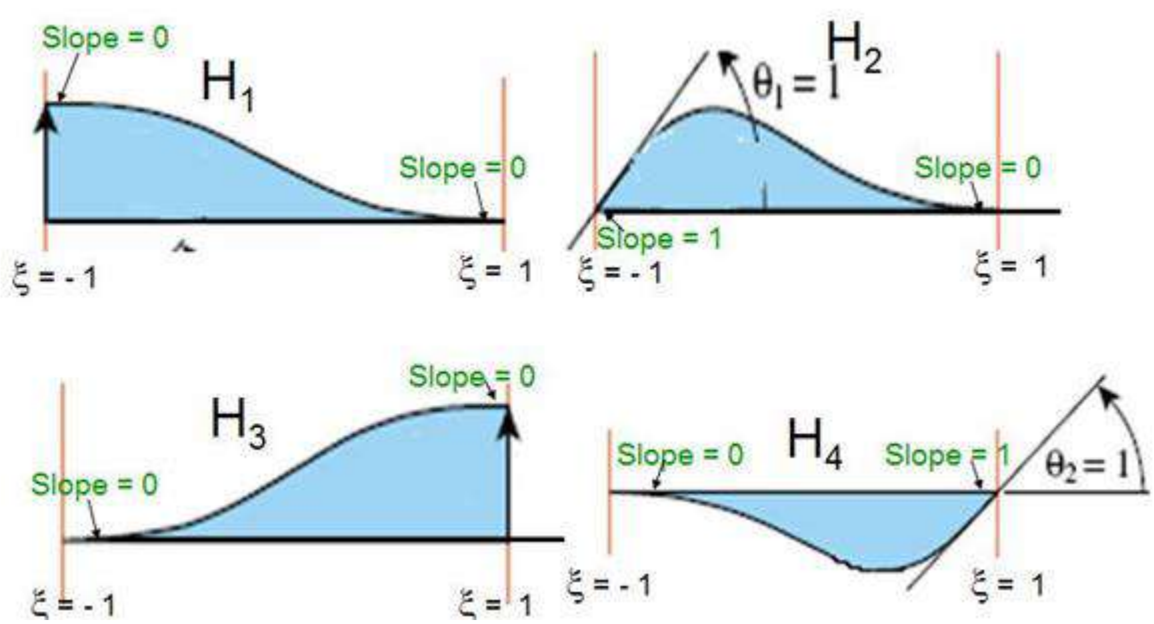
Similarly we can derive

$$H_2 = \tfrac{1}{4}\,(1 - \xi - \xi^2 + \xi^3)$$

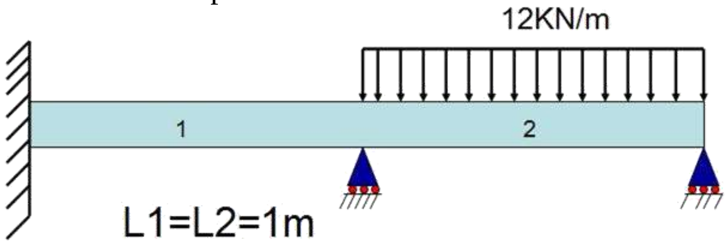
$$H_3 = \tfrac{1}{4}\,(2 + 3\xi - \xi^3)$$

$$H_4 = \tfrac{1}{4}\,(-1 - \xi + \xi^2 + \xi^3)$$

Following graph shows the variations of Hermite shape functions



5. For the beam and loading shown in Fig., determine (1) the slopes at 2 and 3 and (2) the vertical deflection at the midpoint of the distributed load.



L1=L2=1m

E = 200GPa

I = 4 X 10^6N/mm^4

Solution:

Let's model the given system as 2 elements 3 nodes finite element model each node having 2 dof. For each element determine stiffness matrix.

$$K_1 = 8 \text{ X } 10^5 \begin{array}{c c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} & \\ \begin{pmatrix} 12 & 6 & -12 & 6 \\ 6 & 4 & -6 & 2 \\ -12 & -6 & 12 & -6 \\ 6 & 4 & -6 & 4 \end{pmatrix} & \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} \end{array} \quad K_2 = 8 \text{ X } 10^5 \begin{array}{c c} \begin{array}{cccc} 3 & 4 & 5 & 6 \end{array} & \\ \begin{pmatrix} 12 & 6 & -12 & 6 \\ 6 & 4 & -6 & 2 \\ -12 & -6 & 12 & -6 \\ 6 & 4 & -6 & 4 \end{pmatrix} & \begin{array}{c} 3 \\ 4 \\ 5 \\ 6 \end{array} \end{array}$$

Global stiffness matrix

$$K = 8 \text{ X } 10^5 \begin{array}{c c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} & \\ \begin{pmatrix} 12 & 6 & -12 & 6 & 0 & 0 \\ 6 & 4 & -6 & 2 & 0 & 0 \\ -12 & -6 & 24 & 0 & -12 & 6 \\ 6 & 2 & 0 & 8 & -6 & 2 \\ 0 & 0 & -12 & -6 & 12 & -6 \\ 0 & 0 & 6 & 2 & -6 & 4 \end{pmatrix} & \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \end{array}$$

Load vector because of UDL

Element 1 do not contain any UDL hence all the force term for element 1 will be zero.

ie

$$F_1 = \begin{pmatrix} F1 \\ F2 \\ F3 \\ F4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

For element 2 that has UDL its equivalent load and moment are represented as

ie

$$F_2 = \begin{pmatrix} F3 \\ F4 \\ F5 \\ F6 \end{pmatrix} = \begin{pmatrix} -6000 \\ -1000 \\ -6000 \\ 1000 \end{pmatrix}$$

Global load vector:

$$F = \begin{pmatrix} F1 \\ F2 \\ F3 \\ F4 \\ F5 \\ F6 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -6000 \\ -1000 \\ -6000 \\ 1000 \end{pmatrix}$$

From KQ=F we write

$$8 \times 10^5 \begin{pmatrix} 12 & 6 & -12 & 6 & 0 & 0 \\ 6 & 4 & -6 & 2 & 0 & 0 \\ -12 & -6 & 24 & 0 & -12 & 6 \\ 6 & 2 & 0 & 8 & -6 & 2 \\ 0 & 0 & -12 & -6 & 12 & -6 \\ 0 & 0 & 6 & 2 & -6 & 4 \end{pmatrix} \begin{pmatrix} Q1 \\ Q2 \\ Q3 \\ Q4 \\ Q5 \\ Q6 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 6000 \\ -1000 \\ -6000 \\ 1000 \end{pmatrix}$$

At node 1 since its fixed both
q1=q2=0 node 2 because of roller
q3=0 node 3 again roller ie q5= 0
By elimination method the matrix reduces to 2 X 2 solving this we have Q4= -2.679×10^{-4}mm and Q6 = 4.464×10^{-4}mm

To determine the deflection at the middle of element 2 we can write the displacement function as

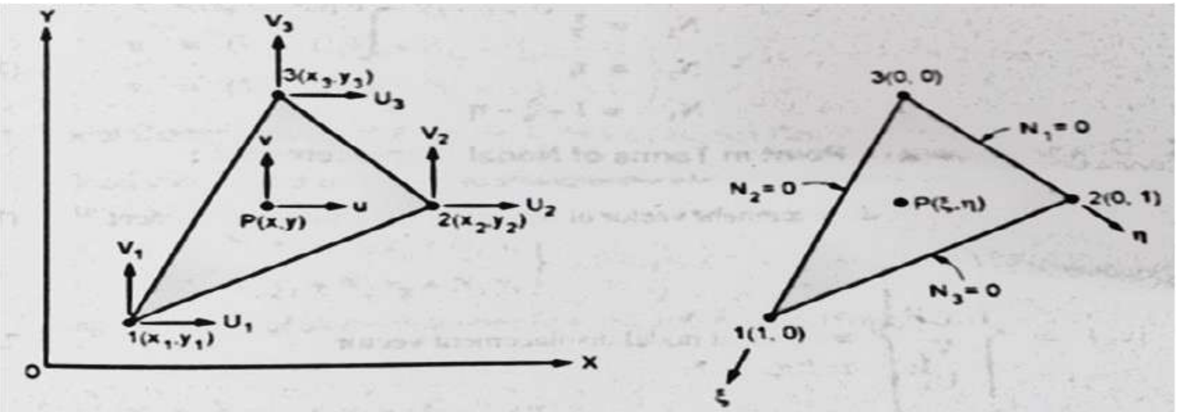
$$V(\xi) = H_1 q_3 + H_2 q_4 \frac{L_e}{2} + H_3 q_5 + H_4 q_6 \frac{L_e}{2}$$

= -0.089mm

UNIT IV

1. Derive the expression for strain displacement matrix and stiffness matrix for a constant strain triangular element.

A: Let us consider a CST as shown in figure



(a)

(a) Local Coordinate System

(b)

(b) Natural Coordinate System

Three-Node Constant Strain Triangular (CST) Element

Let, (x, y) = local coordinates of any point **'P'** within the element

(ξ, η) = natural coordinates of any point **'P'** within the element

1. **Shape Functions:**

- The variation of the different properties such as: displacement, strain, temperature, etc. within the element is interpolated by using the shape functions.
- The three shape functions of any point **'P'** within the element are: N_1, N_2 and N_3.
- **At node 1:** $N_1 = 1, N_2 = 0$, and $N_3 = 0$
- **At node 2:** $N_2 = 1, N_3 = 0$, and $N_1 = 0$
- **At node 1:** $N_3 = 1, N_1 = 0$, and $N_2 = 0$

From figure,

- **At node 1:** $N_1 = 1$ and $\xi = 1$; $N_2 = 0$, and $\eta = 0$.
- **At node 1:** $N_2 = 1$ and $\eta = 1$; $N_1 = 0$, and $\xi = 0$.
- **At node 1:** $N_3 = 1$; $N_1 = 0$ and $\xi = 0$; $N_2 = 0$, and $\eta = 0$.

- Hence, the three shape functions of any point **'P'** within the element are given by,

$$\left.\begin{aligned} \mathbf{N_1} &= \xi \\ \mathbf{N_2} &= \eta \\ \mathbf{N_3} &= \mathbf{1} - \xi - \eta \end{aligned}\right\} \text{------------(1)}$$

2. **Displacement of Point in Terms of Nodal Displacements:**

Let $\{u\} = \begin{Bmatrix} u \\ v \end{Bmatrix}$ = displacement vector of any point **'P'** within the element -----------(2)

$$\{u_N\} = \begin{Bmatrix} U_1 \\ V_1 \\ U_2 \\ V_2 \\ U_3 \\ V_3 \end{Bmatrix} = \text{element nodal displacement vector} \text{ -----------(3)}$$

- Hence, using the shape functions, the displacements of any point **'P'** within the element can be written in terms of the element nodal displacements as,

$$\left.\begin{aligned} \mathbf{u} &= \boldsymbol{N_1U_1 + N_2U_2 + N_3U_3} \\ \mathbf{v} &= \boldsymbol{N_1V_1 + N_2V_2 + N_3V_3} \end{aligned}\right\} \text{-----------(4)}$$

- $$\left.\begin{aligned} \text{or } \mathrm{u} &= (U_1 - U_3)\xi + (U_2 - U_3)\eta + U_3 \\ \mathrm{v} &= (V_1 - V_3)\xi + (V_2 - V_3)\eta + V_3 \end{aligned}\right\} \text{-----------(8)}$$

3. **Local Coordinates of Point in terms of Nodal Coordinates:**

- The local coordinates of any point **P** within the element can be written in terms of shape functions and nodal coordinates as,

$$\left.\begin{aligned} \mathbf{x} &= \boldsymbol{N_1x_1 + N_2x_2 + N_3x_3} \\ \mathbf{y} &= \boldsymbol{N_1y_1 + N_2y_2 + N_3y_3} \end{aligned}\right\} \text{-----------(9)}$$

- Putting the values of shape functions from equation (1) in equation (9),

$$\mathrm{x} = \xi x_1 + \eta x_2 + (1 - \xi - \eta)x_3$$

$$\mathrm{y} = \xi y_1 + \eta y_2 + (1 - \xi - \eta)y_3$$

$$\left.\begin{aligned} \text{or } \mathrm{x} &= (x_1 - x_3)\xi + (x_2 - x_3)\eta + x_3 \\ \mathrm{y} &= (y_1 - y_3)\xi + (y_2 - y_3)\eta + y_3 \end{aligned}\right\} \text{-----------(10)}$$

- Using the notations $\mathrm{x_{ij}} = \mathrm{x_i} - x_j$ and $y_{\mathrm{ij}} = y_{\mathrm{i}} - y_j$, the equations (10) can be written as,

$$\left.\begin{aligned} \mathbf{x} &= \boldsymbol{x_{13}}\xi + \boldsymbol{x_{23}}\eta + \boldsymbol{x_3} \\ \mathbf{y} &= \boldsymbol{y_{13}}\xi + \boldsymbol{y_{23}}\eta + \boldsymbol{y_3} \end{aligned}\right\} \text{-----------(11)}$$

- This equation (11) relates the local coordinates of any point **'P'** with the natural coordinates.
- Equations (4) and (9) are referred as **isoparametric representation** of element.

Element Strain-Nodal Displacement Relationship For CST Element

- In order to determine the strain at any point **'P'** within the element, it is necessary to obtain partial derivatives of **'u'** and **'v'** with respect to **'x'** and **'y'.**
- However, **'u'** and **'v'** are functions of **'ξ'** and **'η'**, as given in equation (8). Using the chain rule for partial derivatives of **u,** we get,

$$\left.\begin{aligned} \frac{\partial u}{\partial \xi} &= \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial \xi} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial \xi} \\ \frac{\partial u}{\partial \eta} &= \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial \eta} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial \eta} \end{aligned}\right\} \quad \text{-----------(12)}$$

- From equations (11),

$$\left.\begin{aligned} \frac{\partial x}{\partial \xi} &= x_{13} \\ \frac{\partial x}{\partial \eta} &= x_{23} \\ \frac{\partial y}{\partial \xi} &= y_{13} \\ \frac{\partial y}{\partial \eta} &= y_{23} \end{aligned}\right\} \quad \text{-----------(13)}$$

- Substituting equations (13) in equation (11),

$$\left.\begin{aligned} \frac{\partial u}{\partial \xi} &= x_{13}\cdot\frac{\partial u}{\partial x} + y_{13}\cdot\frac{\partial u}{\partial y} \\ \frac{\partial u}{\partial \eta} &= x_{23}\cdot\frac{\partial u}{\partial x} + y_{23}\cdot\frac{\partial u}{\partial y} \end{aligned}\right\} \quad \text{-----------(14)}$$

- Above equation (14) can be written in matrix form as,

$$\begin{Bmatrix} \frac{\partial u}{\partial \xi} \\ \frac{\partial u}{\partial \eta} \end{Bmatrix} = \begin{bmatrix} x_{13} & y_{13} \\ x_{23} & y_{23} \end{bmatrix} \begin{Bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \end{Bmatrix} \quad \text{----------(15)}$$

$$\begin{Bmatrix} \frac{\partial u}{\partial \xi} \\ \frac{\partial u}{\partial \eta} \end{Bmatrix} = [J] \begin{Bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \end{Bmatrix} \quad \text{----------(16)}$$

$$[\mathbf{J}] = \begin{bmatrix} \boldsymbol{x_{13}} & \boldsymbol{y_{13}} \\ \boldsymbol{x_{23}} & \boldsymbol{y_{23}} \end{bmatrix} \quad \text{----------(17)}$$

- This matrix [J] is known as **Jacobian of Transformation.**
- From equation (16)

$$\begin{Bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \end{Bmatrix} = [J]^{-1} \begin{Bmatrix} \frac{\partial u}{\partial \xi} \\ \frac{\partial u}{\partial \eta} \end{Bmatrix} \quad \text{----------(18)}$$

- The inverse of Jacobian Transformation [J] is given by,

$$[J]^{-1} = \frac{1}{|J|} \begin{bmatrix} y_{23} & -y_{13} \\ -x_{23} & x_{13} \end{bmatrix} \quad \text{----------(19)}$$

- From equation (8),

$$\left.\begin{aligned} \frac{\partial u}{\partial \xi} &= \{U_1 - U_3\} \\ \frac{\partial u}{\partial \eta} &= \{U_2 - U_3\} \end{aligned}\right\} \quad \text{----------(20)}$$

- Substituting equation (19) and (20) in equation (18), we get,

$$\begin{Bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \end{Bmatrix} = \frac{1}{|J|} \begin{bmatrix} y_{23} & -y_{13} \\ -x_{23} & x_{13} \end{bmatrix} \begin{Bmatrix} U_1 - U_3 \\ U_2 - U_3 \end{Bmatrix}$$

$$\begin{Bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \end{Bmatrix} = \frac{1}{|J|} \begin{Bmatrix} y_{23}(U_1 - U_3) - y_{13}(U_2 - U_3) \\ -x_{23}(U_1 - U_3) + x_{13}(U_2 - U_3) \end{Bmatrix} \quad \text{----------(21)}$$

Similarly,

$$\begin{Bmatrix} \frac{\partial v}{\partial x} \\ \frac{\partial v}{\partial y} \end{Bmatrix} = \frac{1}{|J|} \begin{Bmatrix} y_{23}(V_1 - V_3) - y_{13}(V_2 - V_3) \\ -x_{23}(V_1 - V_3) + x_{13}(V_2 - V_3) \end{Bmatrix} \quad \text{----------(22)}$$

where, $|J| = (x_{13} \cdot y_{23} - x_{23} \cdot y_{13})$

= determinant of [J]

1. **Determination of Element Strain Matrix:**

- For plane two-dimensional element, the components of strains at any point **'P'** within the element are given by,

$$\{\epsilon\} = \begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial v}{\partial y} \\ \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x} \end{Bmatrix} \quad \text{-----------(23)}$$

- Substituting equations (21) and (22) in equation (23), we get,

$$\{\epsilon\} = \frac{1}{|J|}\begin{Bmatrix} y_{23}(U_1 - U_3) - y_{13}(U_2 - U_3) \\ -x_{23}(V_1 - V_3) + x_{13}(V_2 - V_3) \\ -x_{23}(U_1 - U_3) + x_{13}(U_2 - U_3) + y_{23}(V_1 - V_3) - y_{13}(V_2 - V_3) \end{Bmatrix}$$

$$\{\epsilon\} = \frac{1}{|J|}\begin{Bmatrix} y_{23}U_1 - y_{13}U_2 + (y_{13} - y_{23})U_3 \\ -x_{23}V_1 + x_{13}V_2 - (x_{13} - x_{23})V_3 \\ -x_{23}U_1 + y_{23}V_1 + x_{13}U_2 - y_{13}V_2 - (x_{13} - x_{23})U_3 + (y_{13} - y_{23})V_3 \end{Bmatrix} \quad \text{-----------(24)}$$

- But,

$$\left.\begin{aligned} x_{13} - x_{23} &= (x_1 - x_3) - (x_2 - x_3) = x_1 - x_2 = x_{12} \\ y_{13} - y_{23} &= (y_1 - y_3) - (y_2 - y_3) = y_1 - y_2 = y_{12} \end{aligned}\right\} \quad \text{-----------(25)}$$

- Substituting equations (25) in (24), we get

$$\{\epsilon\} = \frac{1}{|J|}\begin{Bmatrix} y_{23}U_1 - y_{13}U_2 + y_{12}U_3 \\ -x_{23}V_1 + x_{13}V_2 - x_{12}V_3 \\ -x_{23}U_1 + y_{23}V_1 + x_{13}U_2 - y_{13}V_2 - x_{12}U_3 + y_{12}V_3 \end{Bmatrix}$$

$$= \frac{1}{|J|}\begin{Bmatrix} y_{23}U_1 + 0 \cdot V_1 - y_{13}U_2 + 0 \cdot V_2 + y_{12}U_3 + 0 \cdot V_3 \\ 0 \cdot U_1 - x_{23}V_1 + 0 \cdot U_2 + x_{13}V_2 + 0 \cdot U_3 - x_{12}V_3 \\ -x_{23}U_1 + y_{23}V_1 + x_{13}U_2 - y_{13}V_2 - x_{12}U_3 + y_{12}V_3 \end{Bmatrix}$$

$$\{\epsilon\} = \frac{1}{|J|}\begin{bmatrix} y_{23} & 0 & -y_{13} & 0 & y_{12} & 0 \\ 0 & -x_{23} & 0 & x_{13} & 0 & -x_{12} \\ -x_{23} & y_{23} & x_{13} & -y_{13} & -x_{12} & y_{12} \end{bmatrix}\begin{Bmatrix} U_1 \\ V_1 \\ U_2 \\ V_2 \\ U_3 \\ V_3 \end{Bmatrix} \text{ -----------(26)}$$

$$\{\epsilon\} = [B]\{u_N\} \text{ -----------(27)}$$

where,

$$\{\epsilon\} = \begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{Bmatrix} = \textbf{element strain vector}$$

$$[\mathbf{B}] = \frac{1}{|J|}\begin{bmatrix} y_{23} & 0 & -y_{13} & 0 & y_{12} & 0 \\ 0 & -x_{23} & 0 & x_{13} & 0 & -x_{12} \\ -x_{23} & y_{23} & x_{13} & -y_{13} & -x_{12} & y_{12} \end{bmatrix}$$

$$[\mathbf{B}] = \frac{1}{|J|}\begin{bmatrix} y_{23} & 0 & y_{31} & 0 & y_{12} & 0 \\ 0 & x_{32} & 0 & x_{13} & 0 & x_{21} \\ x_{32} & y_{23} & x_{13} & y_{31} & x_{21} & y_{12} \end{bmatrix}$$

$$= \textbf{element strain nodal displacement matrix}$$

1. Element Stiffness Matrix

We know,

$$[\mathbf{k}]\{u_N\} = \{\mathbf{f}\}$$

where,

$$[\mathbf{k}] = \mathbf{t}\int_e [B]^T[D][B]dA \text{ -----------(29)}$$

$$= \textbf{element stiffness matrix}$$

where,

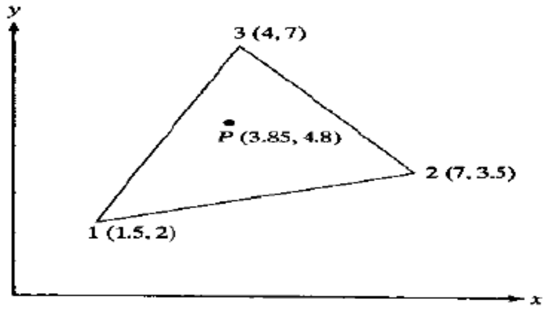
$$\int_e dA = A$$

A = area of the element

$$\text{A} = \frac{1}{2}(\text{Magnitude of } |\text{J}|)$$

$$[\mathbf{k}] = \mathbf{t}\mathbf{A}[B]^T[D][B] \text{ -----------(30)}$$

2. Evaluate the shape functions N_1, N_2 and N_3 at the interior point *P* for the triangular element shown in Fig.



FIGURE

Solution Using the isoparametric representation we have

$$3.85 = 1.5N_1 + 7N_2 + 4N_3 = -2.5\xi + 3\eta + 4$$
$$4.8 = 2N_1 + 3.5N_2 + 7N_3 = -5\xi - 3.5\eta + 7$$

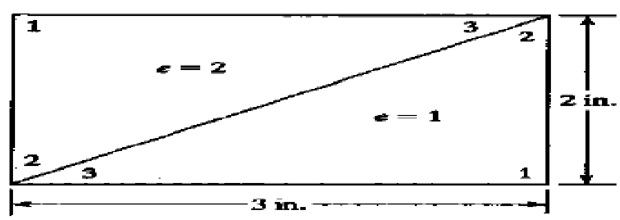
These two equations are rearranged in the form

$$2.5\xi - 3\eta = 0.15$$
$$5\xi + 3.5\eta = 2.2$$

Solving the equations, we obtain $\xi = 0.3$ and $\eta = 0.2$, which implies that

$$N_1 = 0.3 \qquad N_2 = 0.2 \qquad N_3 = 0.5$$

3. Find the strain-nodal displacement matrix B' for the elements shown in Fig.. Use local numbers given at the corners.



FIGURE

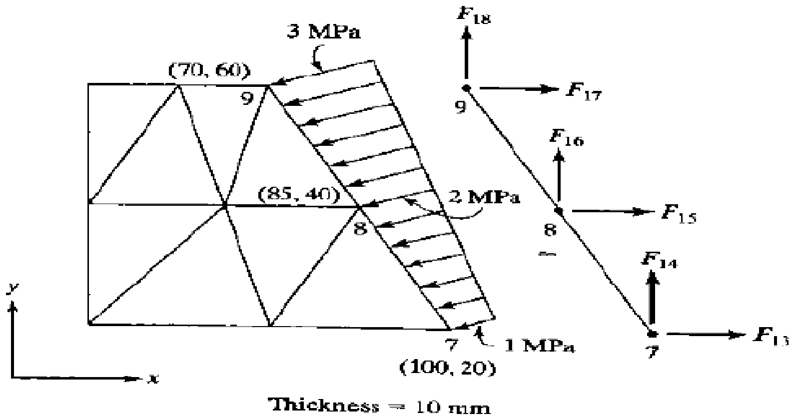
Solution We have

$$\mathbf{B}^1 = \frac{1}{\det \mathbf{J}}\begin{bmatrix} y_{23} & 0 & y_{31} & 0 & y_{12} & 0 \\ 0 & x_{32} & 0 & x_{13} & 0 & x_{21} \\ x_{32} & y_{23} & x_{13} & y_{31} & x_{21} & y_{12} \end{bmatrix}$$
$$= \frac{1}{6}\begin{bmatrix} 2 & 0 & 0 & 0 & -2 & 0 \\ 0 & -3 & 0 & 3 & 0 & 0 \\ -3 & 2 & 3 & 0 & 0 & -2 \end{bmatrix}$$

where $\det \mathbf{J}$ is obtained from $x_{13}y_{23} - x_{23}y_{13} = (3)(2) - (3)(0) = 6$. Using the local numbers at the corners, $\mathbf{B}^2$ can be written using the relationship as

$$\mathbf{B}^2 = \frac{1}{6}\begin{bmatrix} -2 & 0 & 0 & 0 & 2 & 0 \\ 0 & 3 & 0 & -3 & 0 & 0 \\ 3 & -2 & -3 & 0 & 0 & 2 \end{bmatrix}$$

4. A two-dimensional plate is shown in the Fig. Determine the equivalent point loads at nodes 7,8, and 9 for the linearly distributed pressure load acting on the edge 7-8-9.



FIGURE

Solution We consider the two edges 7–8 and 8–9 separately and then merge them.

For edge 7–8

$p_1 = 1$ MPa, $p_2 = 2$ MPa, $x_1 = 100$ mm, $y_1 = 20$ mm, $x_2 = 85$ mm, $y_2 = 40$ mm.

$$\ell_{1-2} = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} = 25 \text{ mm}$$

$$c = \frac{y_2 - y_1}{\ell_{1-2}} = 0.8, \qquad s = \frac{x_1 - x_2}{\ell_{1-2}} = 0.6$$

$$T_{x1} = -p_1 c = -0.8, \qquad T_{y1} = -p_1 s = -0.6, \qquad T_{x2} = -p_2 c = -1.6,$$

$$T_{y2} = -p_2 s = -1.2$$

$$\mathbf{T}^1 = \frac{10 \times 25}{6}[2T_{x1} + T_{x2}, 2T_{y1} + T_{y2}, T_{x1} + 2T_{x2}, T_{y1} + 2T_{y2}]^T$$

$$= [-133.3, -100, -166.7, -125]^T \text{ N}$$

These loads add to F_{13}, F_{14}, F_{15}, and F_{16}, respectively.

For edge 8–9

$p_1 = 2$ MPa, $p_2 = 3$ MPa, $x_1 = 85$ mm, $y_1 = 40$ mm, $x_2 = 70$ mm, $y_2 = 60$ mm,

$$\ell_{1-2} = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} = 25 \text{ mm}$$

$$c = \frac{y_2 - y_1}{\ell_{1-2}} = 0.8, \qquad s = \frac{x_1 - x_2}{\ell_{1-2}} = 0.6$$

$$T_{x1} = -p_1 c = -1.6, \qquad T_{y1} = -p_1 s = -1.2, \qquad T_{x2} = -p_2 c = -2.4,$$

$$T_{y2} = -p_2 s = -1.8$$

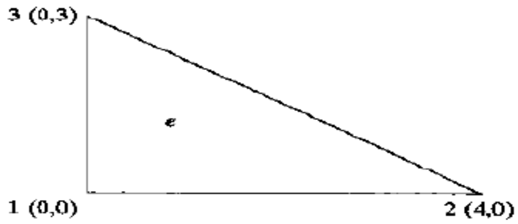
$$\mathbf{T}^2 = \frac{10 \times 25}{6}[2T_{x1} + T_{x2}, 2T_{y1} + T_{y2}, T_{x1} + 2T_{x2}, T_{y1} + 2T_{y2}]^T$$

$$= [-233.3, -175, -266.7, -200]^T \text{ N}$$

These loads add to F_{15}, F_{16}, F_{17}, and F_{18}, respectively. Thus,

$$[F_{13} \quad F_{14} \quad F_{15} \quad F_{16} \quad F_{17} \quad F_{18}] = [-133.3 \quad -100 \quad -400 \quad -300 \quad -266.7 \quad -200] \text{ N}$$

5. A CST element is shown in Fig. The element is subjected to a body force $f_x = x^2$ N/m^3. Determine the nodal force vector f^e. Take element thickness = 1 m.



FIGURE

The work potential is $-\int_e \mathbf{f}^T\mathbf{u}\,dV$, where $\mathbf{f}^T = [f_x, 0]$. Substituting for $\mathbf{u} = \mathbf{N}\mathbf{q}$, we obtain the work potential in the form $-\mathbf{q}^T\mathbf{f}^e$, where $\mathbf{f}^e = \int_e \mathbf{N}^T\mathbf{f}\,dV$, where $\mathbf{N}$ is given

All y components of $\mathbf{f}^e$ are zero. The x components at nodes 1, 2, 3 are given, respectively, by

$$\int_e \xi f_x\,dV, \quad \int_e \eta f_x\,dV, \quad \int_e (1-\xi-\eta) f_x\,dV$$

We now make the following substitutions: $f_x = x^2$, $x = \xi x_1 + \eta x_2 + (1-\xi-\eta)x_3 = 4\eta$, $dV = \det \mathbf{J}\,d\eta\,d\xi$, $\det \mathbf{J} = 2A_e$, and $A_e = 6$. Now, integration over a triangle is illustrated in Fig. 5.6. Thus,

$$\int_e \xi f_x\,dV = \int_0^1 \int_0^{1-\xi} \xi(16\eta^2)(12)\,d\eta\,d\xi = 3.2\text{ N}$$

Similarly, the other integrations result in 9.6 N and 3.2 N. Thus,

$$\mathbf{f}^e = [3.2, 0, 9.6, 0, 3.2, 0]^T\text{ N}$$

6. For the two-dimensional loaded plate shown in FIg, determine the displacements of nodes 1 and 2 and the element stresses using plane stress conditions. Body force may be neglected in comparison with the external forces.

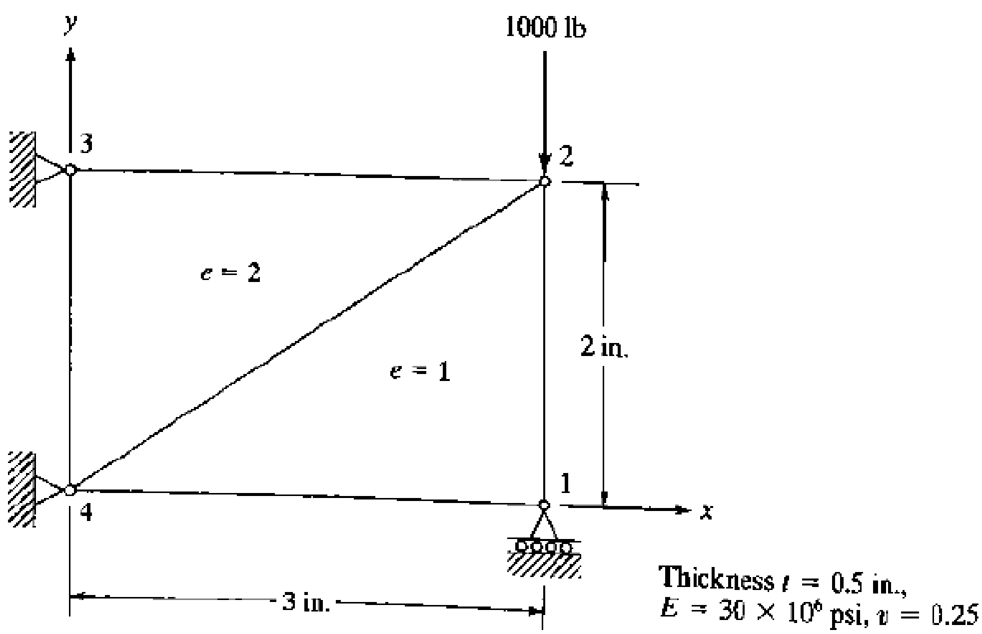


FIGURE 1

Solution For plane stress conditions, the material property matrix is given by

$$\mathbf{D} = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \dfrac{1-\nu}{2} \end{bmatrix} = \begin{bmatrix} 3.2 \times 10^7 & 0.8 \times 10^7 & 0 \\ 0.8 \times 10^7 & 3.2 \times 10^7 & 0 \\ 0 & 0 & 1.2 \times 10^7 \end{bmatrix}$$

Using the local numbering pattern used in Fig. E5.3, we establish the connectivity as follows:

	Nodes		
Element No.	1	2	3
1	1	2	4
2	3	4	2

On performing the matrix multiplication $\mathbf{DB}^e$, we get

$$\mathbf{DB}^1 = 10^7\begin{bmatrix} 1.067 & -0.4 & 0 & 0.4 & -1.067 & 0 \\ 0.267 & -1.6 & 0 & 1.6 & -0.267 & 0 \\ -0.6 & 0.4 & 0.6 & 0 & 0 & -0.4 \end{bmatrix}$$

and

$$\mathbf{DB}^2 = 10^7\begin{bmatrix} -1.067 & 0.4 & 0 & -0.4 & 1.067 & 0 \\ -0.267 & 1.6 & 0 & -1.6 & 0.267 & 0 \\ 0.6 & -0.4 & -0.6 & 0 & 0 & 0.4 \end{bmatrix}$$

These two relationships will be used later in calculating stresses using $\boldsymbol{\sigma}^e = \mathbf{DB}^e\mathbf{q}$. The multiplication $t_eA_e\mathbf{B}^{e^T}\mathbf{DB}^e$ gives the element stiffness matrices,

$$\begin{array}{cc} & \begin{array}{cccccc} 1 & 2 & 3 & 4 & 7 & 8 \leftarrow \text{Global dof} \end{array} \\ \mathbf{k}^1 = 10^7 & \begin{bmatrix} 0.983 & -0.5 & -0.45 & 0.2 & -0.533 & 0.3 \\ & 1.4 & 0.3 & -1.2 & 0.2 & -0.2 \\ & & 0.45 & 0 & 0 & -0.3 \\ & & & 1.2 & -0.2 & 0 \\ & \text{Symmetric} & & & 0.533 & 0 \\ & & & & & 0.2 \end{bmatrix} \end{array}$$

$$\begin{array}{cc} & \begin{array}{cccccc} 5 & 6 & 7 & 8 & 3 & 4 \leftarrow \text{Global dof} \end{array} \\ \mathbf{k}^2 = 10^7 & \begin{bmatrix} 0.983 & -0.5 & -0.45 & 0.2 & -0.533 & 0.3 \\ & 1.4 & 0.3 & -1.2 & 0.2 & -0.2 \\ & & 0.45 & 0 & 0 & -0.3 \\ & & & 1.2 & -0.2 & 0 \\ & \text{Symmetric} & & & 0.533 & 0 \\ & & & & & 0.2 \end{bmatrix} \end{array}$$

In the previous element matrices, the global dof association is shown on top. In the problem under consideration, Q_2, Q_5, Q_6, Q_7, and Q_8, are all zero. Using the elimination approach discussed in , it is now sufficient to consider the stiffnesses associated with the degrees of freedom Q_1, Q_3, and Q_4. Since the body forces are neglected, the first vector has the component $F_4 = -1000$ lb. The set of equations is given by the matrix representation

$$10^7\begin{bmatrix} 0.983 & -0.45 & 0.2 \\ -0.45 & 0.983 & 0 \\ 0.2 & 0 & 1.4 \end{bmatrix}\begin{Bmatrix} Q_1 \\ Q_3 \\ Q_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ -1000 \end{Bmatrix}$$

Solving for Q_1, Q_3, and Q_4, we get

$$Q_1 = 1.913 \times 10^{-5}\text{ in.} \qquad Q_3 = 0.875 \times 10^{-5}\text{ in.} \qquad Q_4 = -7.436 \times 10^{-5}\text{ in.}$$

For element 1, the element nodal displacement vector is given by

$$\mathbf{q}^1 = 10^{-5}[1.913, 0, 0.875, -7.436, 0, 0]^T$$

The element stresses $\boldsymbol{\sigma}^1$ are calculated from $\mathbf{DB}^1\mathbf{q}$ as

$$\boldsymbol{\sigma}^1 = [-93.3, -1138.7, -62.3]^T\text{ psi}$$

Similarly,

$$\mathbf{q}^2 = 10^{-5}[0, 0, 0, 0, 0.875, -7.436]^T$$
$$\boldsymbol{\sigma}^2 = [93.4, 23.4, -297.4]^T\text{ psi}$$

7.

A long hollow cylinder of inside diameter 100 mm and outside diameter 120 mm is firmly fitted in a hole of another rigid cylinder over its full length as shown in figure 8.14. The cylinder is then subjected to an internal pressure of 2 MPa. By using two elements on the 10 mm length shown, find the displacements at the inner radius. Take E = 210 GPa; $\mu = 0.25$

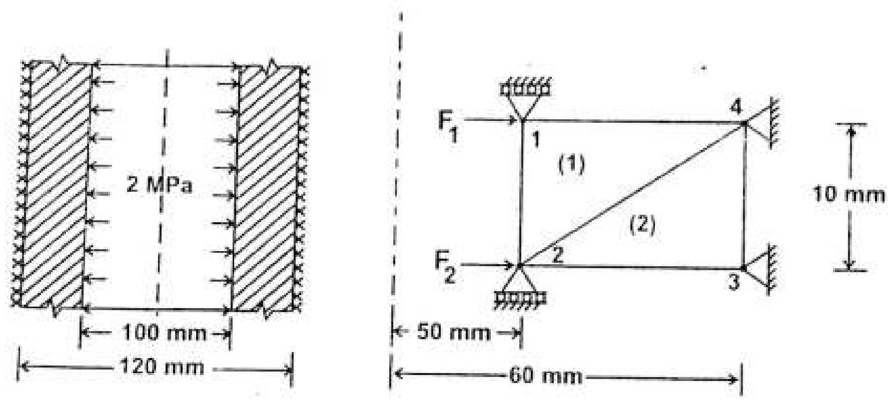


Fig. 8.14

Solution:

Since the portion of interest (i.e., the 10 mm length part) contains two elements (1) & (2) of triangular shape as shown, we must evaluate the global stiffness matrix in order to determine the displacements in the required points (i.e., at the inner nodes 1 & 2)

Now, the connectivity (i.e., the relationship) between the local element nodes and global nodes are given along with their coordinates as follows.

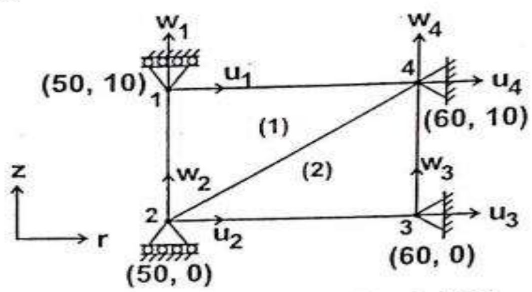
Connectivity

Element nodes	1	2	3
Global nodes			
For element (1)	1	2	4
For element (2)	2	3	4

Coordinates in mm

Nodes	r	z
1	50	10
2	50	0
3	60	0
4	60	10

The element with their coordinates and nodal displacements are shown in **fig. 8.14(a)**



E = 210 GPa
= 2.1×10^5 N/mm^2
$\mu = 0.25$

Fig. 8.14(a)

Now consider element (1):

The nodal coordinates as per local element scheme

$r_1 = 50$ mm; $\quad z_1 = 10$ mm

$r_2 = 50$ mm; $\quad z_2 = 0$ mm

$r_3 = 60$ mm; $\quad z_3 = 10$ mm

(Note: The node 3 of element (1) is actually meant for node 4.)

For axisymmetric triangular element (1) the stiffness matrix is given by the expression as

$$[k]_1 = 2\,\pi\, rA\ [B]^T\ [D]\ [B] \qquad \text{.... (1)}$$

where,

A = Area of the triangle

r = Common coordinate of the element

[D] = Stress-strain relationship matrix

[B] = Strain-displacement matrix.

Now,

$$\text{Area of triangle, } A = \frac{1}{2}\begin{vmatrix} 1 & r_1 & z_1 \\ 1 & r_2 & z_2 \\ 1 & r_3 & z_3 \end{vmatrix} = \frac{1}{2}\begin{vmatrix} 1 & 50 & 10 \\ 1 & 50 & 0 \\ 1 & 60 & 10 \end{vmatrix}$$

$$= \frac{1}{2}\,[1\,(500-0) - 50\,(10-0) + 10\,(60-50)] = 50 \text{ mm}^2 \qquad \text{.... (2)}$$

The strain-displacement matrix,

$$[B] = \frac{1}{2A}\begin{bmatrix} \beta_1 & 0 & \beta_2 & 0 & \beta_3 & 0 \\ \left(\dfrac{\alpha_1 + \beta_1 r + \gamma_1 z}{r}\right) & 0 & \left(\dfrac{\alpha_2 + \beta_2 r + \gamma_2 z}{r}\right) & 0 & \left(\dfrac{\alpha_3 + \beta_3 r + \gamma_3 z}{r}\right) & 0 \\ 0 & \gamma_1 & 0 & \gamma_2 & 0 & \gamma_3 \\ \gamma_1 & \beta_1 & \gamma_2 & \beta_2 & \gamma_3 & \beta_3 \end{bmatrix}$$

Now,

$$\alpha_1 = r_2 z_3 - r_3 z_2 = (50 \times 10) - (60 \times 0) = 500$$

$$\alpha_2 = r_3 z_1 - r_1 z_3 = (60 \times 10) - (50 \times 10) = 100$$

$$\alpha_3 = r_1 z_2 - r_2 z_1 = (50 \times 0) - (50 \times 10) = -500$$

$$\beta_1 = z_2 - z_3 = 0 - 10 = -10$$

$$\beta_2 = z_3 - z_1 = 10 - 10 = 0$$

$$\beta_3 = z_1 - z_2 = 10 - 0 = 10$$

$$\gamma_1 = r_3 - r_2 = 60 - 50 = 10$$

$$\gamma_2 = r_1 - r_3 = 50 - 60 = -10$$

$$\gamma_3 = r_2 - r_1 = 50 - 50 = 0$$

The coordinates,

$$r = \frac{r_1 + r_2 + r_3}{3} = \frac{50 + 50 + 60}{3} = \frac{160}{3}$$

$$z = \frac{z_1 + z_2 + z_3}{3} = \frac{10 + 0 + 10}{3} = \frac{20}{3}$$

and the terms

$$\left(\frac{\alpha_1 + \beta_1 r + \gamma_1 z}{r}\right) = \frac{3}{160}\left[500 - 10\left(\frac{160}{3}\right) + 10\left(\frac{20}{3}\right)\right] = 0.625$$

$$\left(\frac{\alpha_2 + \beta_2 r + \gamma_2 z}{r}\right) = \frac{3}{160}\left[100 + 0 - 10\left(\frac{20}{3}\right)\right] = 0.625$$

$$\left(\frac{\alpha_3 + \beta_3 r + \gamma_3 z}{r}\right) = \frac{3}{160}\left[-500 + 10\left(\frac{160}{3}\right) + 0\right] = 0.625$$

Substituting the above values in the strain-displacement matrix, we get

$$[B] = \frac{1}{100}\begin{bmatrix} -10 & 0 & 0 & 0 & 10 & 0 \\ 0.625 & 0 & 0.625 & 0 & 0.625 & 0 \\ 0 & 10 & 0 & -10 & 0 & 0 \\ 10 & -10 & -10 & 0 & 0 & 10 \end{bmatrix} \qquad \text{.... (3)}$$

Hence,

$$[B]^T = \frac{1}{100}\begin{bmatrix} -10 & 0.625 & 0 & 10 \\ 0 & 0 & 10 & -10 \\ 0 & 0.625 & 0 & -10 \\ 0 & 0 & -10 & 0 \\ 10 & 0.625 & 0 & 0 \\ 0 & 0 & 0 & 10 \end{bmatrix} \qquad \text{.... (4)}$$

The stress-strain matrix,

$$[D] = \frac{E}{(1+\mu)(1-2\mu)} \begin{bmatrix} (1-\mu) & \mu & \mu & 0 \\ \mu & (1-\mu) & \mu & 0 \\ \mu & \mu & (1-\mu) & 0 \\ 0 & 0 & 0 & \left(\frac{1-2\mu}{2}\right) \end{bmatrix}$$

$$= \frac{2.1 \times 10^5}{(1+0.25)\,[1-(2 \times 0.25)]} \begin{bmatrix} (1-0.25) & 0.25 & 0.25 & 0 \\ 0.25 & (1-0.25) & 0.25 & 0 \\ 0.25 & 0.25 & (1-0.25) & 0 \\ 0 & 0 & 0 & \left(\frac{1-2 \times 0.25}{2}\right) \end{bmatrix}$$

$$= \frac{2.1 \times 10^5}{(1.25)\,(0.5)} \begin{bmatrix} 0.75 & 0.25 & 0.25 & 0 \\ 0.25 & 0.75 & 0.25 & 0 \\ 0.25 & 0.25 & 0.75 & 0 \\ 0 & 0 & 0 & 0.25 \end{bmatrix}$$

$$= 84 \times 10^3 \begin{bmatrix} 3 & 1 & 1 & 0 \\ 1 & 3 & 1 & 0 \\ 1 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \text{.... (5)}$$

Substituting the values of Eqns. (2), (3), (4) and (5) in Eqn. (1), we get the stiffness matrix for element (1)

i.e., $[k]_1 = 2\,\pi\, r\, A\, [B]^T\, [D]\, [B]$

$$= 2\,\pi \times \frac{160}{3} \times 50 \times \frac{1}{100} \begin{bmatrix} -10 & 0.625 & 0 & 10 \\ 0 & 0 & 10 & -10 \\ 0 & 0.625 & 0 & -10 \\ 0 & 0 & -10 & 0 \\ 10 & 0.625 & 0 & 0 \\ 0 & 0 & 0 & 10 \end{bmatrix} \times$$

$$84 \times 10^3 \begin{bmatrix} 3 & 1 & 1 & 0 \\ 1 & 3 & 1 & 0 \\ 1 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \times \frac{1}{100} \begin{bmatrix} -10 & 0 & 0 & 0 & 10 & 0 \\ 0.625 & 0 & 0.625 & 0 & 0.625 & 0 \\ 0 & 10 & 0 & -10 & 0 & 0 \\ 10 & -10 & -10 & 0 & 0 & 10 \end{bmatrix}$$

$$= 1.4 \times 10^5 \begin{bmatrix} -10 & 0.625 & 0 & 10 \\ 0 & 0 & 10 & -10 \\ 0 & 0.625 & 0 & -10 \\ 0 & 0 & -10 & 0 \\ 10 & 0.625 & 0 & 0 \\ 0 & 0 & 0 & 10 \end{bmatrix} \times$$

$$\begin{bmatrix} -29.375 & 10 & 0.625 & -10 & 30.625 & 0 \\ -8.125 & 10 & 1.875 & -10 & 11.875 & 0 \\ -9.375 & 30 & 0.625 & -30 & 10.625 & 0 \\ 10 & -10 & -10 & 0 & 0 & 10 \end{bmatrix}$$

$$= 1.4 \times 10^5 \begin{bmatrix} 388.67 & -193.75 & -105.08 & 93.75 & -298.83 & 100 \\ -193.75 & 400 & 106.25 & -300 & 106.25 & -100 \\ -105.08 & 106.25 & 101.18 & -6.25 & 74.22 & -100 \\ 93.75 & -300 & -6.25 & 300 & -106.25 & 0 \\ -298.83 & 106.25 & 74.22 & -106.25 & 313.67 & 0 \\ 100 & -100 & -100 & 0 & 0 & 100 \end{bmatrix}$$

$$\text{i.e., } [k]_1 = 10^5 \begin{array}{c} \begin{array}{cccccc} u_1 & w_1 & u_2 & w_2 & u_4 & w_4 \end{array} \\ \begin{bmatrix} 544.1 & -271.3 & -147.1 & 131.3 & -418.4 & 140 \\ -271.3 & 560 & 148.8 & -420 & 148.8 & -140 \\ -147.1 & 148.8 & 141.7 & -8.75 & 103.9 & -140 \\ 131.3 & -420 & -8.75 & 420 & -148.8 & 0 \\ -418.4 & 148.8 & 103.9 & -148.8 & 439.1 & 0 \\ 140 & -140 & -140 & 0 & 0 & 140 \end{bmatrix} \end{array} \begin{array}{c} u_1 \\ w_1 \\ u_2 \\ w_2 \\ u_4 \\ w_4 \end{array} \quad \ldots (6)$$

Similarly consider the element (2)

The coordinates of element (2) in mm are

$r_1 = 50; \quad z_1 = 0$

$r_2 = 60; \quad z_2 = 0$

$r_3 = 60; \quad z_3 = 10$

The stiffness matrix for element (2) is given by,

$$[k]_2 = 2\,\pi\, r\, A\, [B]^T\, [D]\, [B] \quad \ldots (7)$$

$$\text{Now, } A = \frac{1}{2}\begin{vmatrix} 1 & r_1 & z_1 \\ 1 & r_2 & z_2 \\ 1 & r_3 & z_3 \end{vmatrix} = \frac{1}{2}\begin{vmatrix} 1 & 50 & 0 \\ 1 & 60 & 0 \\ 1 & 60 & 10 \end{vmatrix} = 50 \text{ mm}^2 \quad \ldots (8)$$

$$[B] = \frac{1}{2A}\begin{bmatrix} \beta_1 & 0 & \beta_2 & 0 & \beta_3 & 0 \\ \left(\frac{\alpha_1 + \beta_1 r + \gamma_1 z}{r}\right) & 0 & \left(\frac{\alpha_2 + \beta_2 r + \gamma_2 z}{r}\right) & 0 & \left(\frac{\alpha_3 + \beta_3 r + \gamma_3 z}{r}\right) & 0 \\ 0 & \gamma_1 & 0 & \gamma_2 & 0 & \gamma_3 \\ \gamma_1 & \beta_1 & \gamma_2 & \beta_2 & \gamma_3 & \beta_3 \end{bmatrix}$$

$\alpha_1 = r_2 z_3 - r_3 z_2 = (60 \times 10) - (60 \times 0) = 600$

$\alpha_2 = r_3 z_1 - r_1 z_3 = (60 \times 0) - (50 \times 10) = -500$

$\alpha_3 = r_1 z_2 - r_2 z_1 = (50 \times 0) - (60 \times 0) = 0$

$\beta_1 = z_2 - z_3 = 0 - 10 = -10$

$\beta_2 = z_3 - z_1 = 10 - 0 = 10$

$\beta_3 = z_1 - z_2 = 0 - 0 = 0$

$\gamma_1 = r_3 - r_2 = 60 - 60 = 0$

$\gamma_2 = r_1 - r_3 = 50 - 60 = -10$

$\gamma_3 = r_2 - r_1 = 60 - 50 = 10$

The coordinates,

$$r = \frac{r_1 + r_2 + r_3}{3} = \frac{50 + 60 + 60}{3} = \frac{170}{3}$$

$$z = \frac{z_1 + z_2 + z_3}{3} = \frac{0 + 0 + 10}{3} = \frac{10}{3}$$

The terms,

$$\frac{\alpha_1 + \beta_1 r + \gamma_1 z}{r} = \frac{3}{170}\left[600 - 10\left(\frac{170}{3}\right) + 0\right] = 0.59$$

$$\frac{\alpha_2 + \beta_2 r + \gamma_2 z}{r} = \frac{3}{170}\left[-500 + 10\left(\frac{170}{3}\right) - 10\left(\frac{10}{3}\right)\right] = 0.59$$

$$\frac{\alpha_3 + \beta_3 r + \gamma_3 z}{r} = \frac{3}{170}\left[0 + 0 + 10\left(\frac{10}{3}\right)\right] = 0.59$$

Now,

$$[B] = \frac{1}{100}\begin{bmatrix} -10 & 0 & 10 & 0 & 0 & 0 \\ 0.59 & 0 & 0.59 & 0 & 0.59 & 0 \\ 0 & 0 & 0 & -10 & 0 & 10 \\ 0 & -10 & -10 & 10 & 10 & 0 \end{bmatrix} \quad \text{.... (9)}$$

Hence

$$[B]^T = \frac{1}{100}\begin{bmatrix} -10 & 0.59 & 0 & 0 \\ 0 & 0 & 0 & -10 \\ 10 & 0.59 & 0 & -10 \\ 0 & 0 & -10 & 10 \\ 0 & 0.59 & 0 & 10 \\ 0 & 0 & 10 & 0 \end{bmatrix} \quad \text{.... (10)}$$

The stress-strain matrix,

$$[D] = \frac{E}{(1+\mu)(1-2\mu)}\begin{bmatrix} (1-\mu) & \mu & \mu & 0 \\ \mu & (1-\mu) & \mu & 0 \\ \mu & \mu & (1-\mu) & 0 \\ 0 & 0 & 0 & \left(\frac{1-2\mu}{2}\right) \end{bmatrix}$$

$$= \frac{2.1 \times 10^5}{(1.25)(0.5)}\begin{bmatrix} 0.75 & 0.25 & 0.25 & 0 \\ 0.25 & 0.75 & 0.25 & 0 \\ 0.25 & 0.25 & 0.75 & 0 \\ 0 & 0 & 0 & 0.25 \end{bmatrix}$$

$$= 84 \times 10^3 \begin{bmatrix} 3 & 1 & 1 & 0 \\ 1 & 3 & 1 & 0 \\ 1 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{.... (11)}$$

Substituting the values of Eqns. (8), (9), (10) & (11) in Eqn. (7), we get the stiffness matrix for element (2) as.

$$[k]_2 = 2\pi \times \frac{170}{3} \times 50 \times \frac{1}{100}\begin{bmatrix} -10 & 0.59 & 0 & 0 \\ 0 & 0 & 0 & -10 \\ 10 & 0.59 & 0 & -10 \\ 0 & 0 & -10 & 10 \\ 0 & 0.59 & 0 & 10 \\ 0 & 0 & 10 & 0 \end{bmatrix} \times$$

$$84 \times 10^3 \underset{(4 \times 4)}{\begin{bmatrix} 3 & 1 & 1 & 0 \\ 1 & 3 & 1 & 0 \\ 1 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}} \times \frac{1}{100} \underset{(4 \times 6)}{\begin{bmatrix} -10 & 0 & 10 & 0 & 0 & 0 \\ 0.59 & 0 & 0.59 & 0 & 0.59 & 0 \\ 0 & 0 & 0 & -10 & 0 & 10 \\ 0 & -10 & -10 & 10 & 10 & 0 \end{bmatrix}}$$

$$
= 1.5 \times 10^5 \begin{bmatrix} -10 & 0.59 & 0 & 0 \\ 0 & 0 & 0 & -10 \\ 10 & 0.59 & 0 & -10 \\ 0 & 0 & -10 & 10 \\ 0 & 0.59 & 0 & 10 \\ 0 & 0 & 10 & 0 \end{bmatrix} \begin{bmatrix} -29.41 & 0 & 30.59 & -10 & 0.59 & 10 \\ -8.23 & 0 & 11.77 & -10 & 1.77 & 10 \\ -9.41 & 0 & 10.59 & -30 & 0.59 & 30 \\ 0 & -10 & -10 & 10 & 10 & 0 \end{bmatrix}
$$

$$
= 1.5 \times 10^5 \begin{bmatrix} 289.2 & 0 & -299 & 94.1 & -4.9 & -94.1 \\ 0 & 100 & 100 & -100 & -100 & 0 \\ -299 & 100 & 412.8 & -205.9 & -93.1 & 105.9 \\ 94.1 & -100 & -205.9 & 400 & 94.1 & -300 \\ -4.9 & -100 & -93.1 & 94.1 & 101 & 5.9 \\ -94.1 & 0 & 105.9 & -300 & 5.9 & 300 \end{bmatrix}
$$

$$
\text{i.e., } [k]_2 = 10^5 \begin{array}{c} \begin{array}{cccccc} u_2 & w_2 & u_3 & w_3 & u_4 & w_4 \end{array} \\ \begin{bmatrix} 433.8 & 0 & -448.5 & 141.2 & -7.4 & -141.2 \\ 0 & 150 & 150 & -150 & -150 & 0 \\ -448.5 & 150 & 619.2 & -308.9 & -139.7 & 158.9 \\ 141.2 & -150 & -308.9 & 600 & 141.2 & -450 \\ -7.4 & -150 & -139.7 & 141.2 & 151.5 & 8.9 \\ -141.2 & 0 & 158.9 & -450 & 8.9 & 450 \end{bmatrix} \end{array} \begin{array}{c} \\ u_2 \\ w_2 \\ u_3 \\ w_3 \\ u_4 \\ w_4 \end{array} \qquad \ldots (12)
$$

Combining the stiffness matrices $[k]_1$ and $[k]_2$ as mentioned in Eqn. (6) and (12), we can get the global stiffness matrix as $[K] = [k]_1 + [k]_2$

i.e., $[K] = 10^5$

u_1	w_1	u_2	w_2	u_3	w_3	u_4	w_4	
544.1	−271.3	− 147.1	131.3	0	0	− 418.4	140	u_1
− 271.3	560	148.8	− 420	0	0	148.8	− 140	w_1
− 147.1	148.8	141.7 + 433.8	− 8.75 + 0	− 448.5	141.2	103.9 + − 7.4	− 140 + − 141.2	u_2
131.3	− 420	− 8.75 + 0	420 + 150	150	− 150	− 148.8 + − 150	0 + 0	w_2
0	0	− 448.5	150	619.2	− 308.9	− 139.7	158.9	u_3
0	0	141.2	− 150	− 308.9	600	141.2	− 450	w_3
− 418.4	148.8	103.9 + − 7.4	−148.8 + − 150	− 139.7	141.2	439.1 + 151.5	0 + 8.9	u_4
140	− 140	− 140 + − 141.2	0 + 0	158.9	− 450	0 + 8.9	140 + 450	w_4

$$[K] = 10^5 \begin{array}{cccccccc} u_1 & w_1 & u_2 & w_2 & u_3 & w_3 & u_4 & w_4 \end{array}$$

$$\begin{bmatrix} 544.1 & -271.3 & -147.1 & 131.3 & 0 & 0 & -418.4 & 140 \\ -271.3 & 560 & 148.8 & -420 & 0 & 0 & 148.8 & -140 \\ -147.1 & 148.8 & 575.5 & -8.75 & -448.5 & 141.2 & 96.5 & -281.2 \\ 131.3 & -420 & -8.75 & 570 & 150 & -150 & -298.8 & 0 \\ 0 & 0 & -448.5 & 150 & 619.2 & -308.9 & -139.7 & 158.9 \\ 0 & 0 & 141.2 & -150 & -308.9 & 600 & 141.2 & -450 \\ -418.4 & 148.8 & 96.5 & -298.8 & -139.7 & 141.2 & 590.6 & 8.9 \\ 140 & -140 & -281.2 & 0 & 158.9 & -450 & 8.9 & 590 \end{bmatrix} \quad .... (13)$$

We know that the force displacement relationship is given by

$$\{F\} = [K]\{\delta\} \quad .... (14)$$

where $\{F\}$ = Nodal force vector

$$= \left[F_{1r}, F_{1z}, F_{2r}, F_{2z}, F_{3r}, F_{3z}, F_{4r}, F_{4z} \right]^T$$

$[K]$ = Global stiffness matrix

$\{\delta\}$ = Nodal displacement vector

$$= \left[u_1, w_1, u_2, w_2, u_3, w_3, u_4, w_4 \right]^T$$

The forces acting along the radial direction at the nodes 1 and 2 are

$$F_{1r} = F_{2r} = \frac{2\pi r_i l_e p}{2}$$

where r_i = Inner radius of cylinder = 50 mm.

l_e = Length of finite element = 10 mm.

p = Internal pressure = 2 MPa = 2×10^6 N/m^2 = 2 N/mm^2

$$\therefore F_{1r} = F_{2r} = \frac{2\pi \times 50 \times 10 \times 2}{2} = \mathbf{3141.6\ N}$$

The forces F_{1z}, F_{2z}, F_{3z} and F_{4z} are zero and the forces F_{3r} and F_{4r} are acting as reaction forces.

Now, the Eqn. (14) can be written as

$$10^5 \begin{bmatrix} 544.1 & -271.3 & -147.1 & 131.3 & 0 & 0 & -418.4 & 140 \\ -271.3 & 560 & 148.8 & -420 & 0 & 0 & 148.8 & -140 \\ -147.1 & 148.8 & 575.5 & -8.75 & -448.5 & 141.2 & 96.5 & -281.2 \\ 131.3 & -420 & -8.75 & 570 & 150 & -150 & -298.8 & 0 \\ 0 & 0 & -448.5 & 150 & 619.2 & -308.9 & -139.7 & 158.9 \\ 0 & 0 & 141.2 & -150 & -308.9 & 600 & 141.2 & -450 \\ -418.4 & 148.8 & 96.5 & -298.8 & -139.7 & 141.2 & 590.6 & 8.9 \\ 140 & -140 & -281.2 & 0 & 158.9 & -450 & 8.9 & 590 \end{bmatrix} \begin{Bmatrix} u_1 \\ w_1 \\ u_2 \\ w_2 \\ u_3 \\ w_3 \\ u_4 \\ w_4 \end{Bmatrix} = \begin{Bmatrix} F_{1r} \\ F_{1z} \\ F_{2r} \\ F_{2z} \\ F_{3r} \\ F_{3z} \\ F_{4r} \\ F_{4z} \end{Bmatrix}$$

$$\text{i.e., } 10^5 \begin{bmatrix} 544.1 & -271.3 & -147.1 & 131.3 & 0 & 0 & -418.4 & 140 \\ -271.3 & 560 & 148.8 & -420 & 0 & 0 & 148.8 & -140 \\ -147.1 & 148.8 & 575.5 & -8.75 & -448.5 & 141.2 & 96.5 & -281.2 \\ 131.3 & -420 & -8.75 & 570 & 150 & -150 & -298.8 & 0 \\ 0 & 0 & -448.5 & 150 & 619.2 & -308.9 & -139.7 & 158.9 \\ 0 & 0 & 141.2 & -150 & -308.9 & 600 & 141.2 & -450 \\ -418.4 & 148.8 & 96.5 & -298.8 & -139.7 & 141.2 & 590.6 & 8.9 \\ 140 & -140 & -281.2 & 0 & 158.9 & -450 & 8.9 & 590 \end{bmatrix} \begin{Bmatrix} u_1 \\ 0 \\ u_2 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 3141.6 \\ 0 \\ 3141.6 \\ 0 \\ F_{3r} \\ 0 \\ F_{4r} \\ 0 \end{Bmatrix} \quad \text{.... (15)}$$

Neglecting 2nd, 4th, 5th, 6th, 7th and 8th rows and columns in the stiffness matrix of Eqn. (15), the remaining terms gives the matrix equation such as

$$10^5 \begin{bmatrix} 544.1 & -147.1 \\ -147.1 & 575.5 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} 3141.6 \\ 3141.6 \end{Bmatrix} \quad \text{.... (16)}$$

i.e., $$10^5 (544.1\, u_1 - 147.1\, u_2) = 3141.6 \quad \text{... (17)}$$

$$10^5 (-147.1\, u_1 + 575.5\, u_2) = 3141.6 \quad \text{... (18)}$$

$$\text{Eqn. (18)} \times \left(\frac{544.1}{147.1}\right) \Rightarrow 10^5\,(-544.1\,u_1 + 2128.7\,u_2) = 11620.3 \qquad (19$$

$$\text{Eqn. (17)} + (19) \Rightarrow 10^5\,(1981.6\,u_2) = 14762$$

$$\therefore\ u_2 = \frac{14762}{10^5 \times 1981.6} = \mathbf{7.4 \times 10^{-5}}\ \text{mm}$$

$$\text{Eqn. (17)} \Rightarrow u_1 = \frac{1}{544.1}\left(\frac{3141.6}{10^5} + 147.1\,u_2\right)$$

$$= \frac{1}{544.1}\left(3141.6 \times 10^{-5} + 147.1 \times 7.4 \times 10^{-5}\right)$$

$$= \mathbf{7.8 \times 10^{-5}\ mm}$$

Result:

The displacements at the inner radius are

$$\mathbf{u_1 = 7.8 \times 10^{-5}\ mm}$$

$$\mathbf{u_2 = 7.4 \times 10^{-5}\ mm.}$$

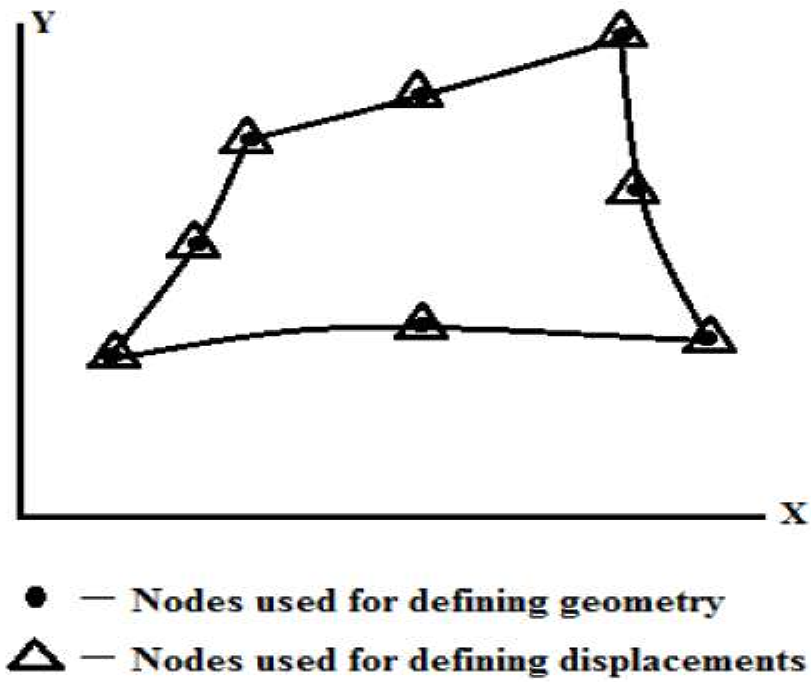
UNIT V

1. Explain the sub-parametric, iso-parametric and super-parametric element.

ISOPARAMETRIC ELEMENT

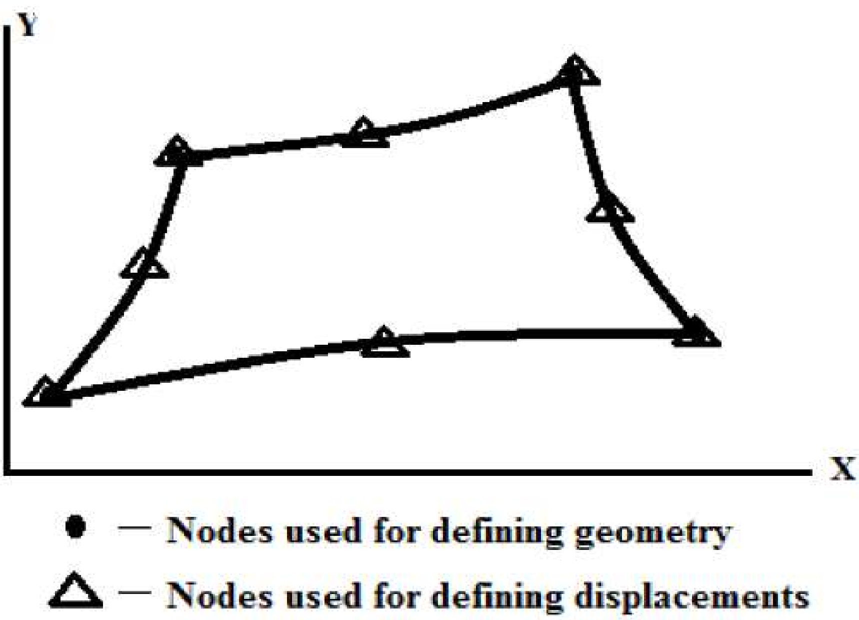
Generally it is very difficult to represent the curved boundaries by straight edge elements. A large number of elements may be used to obtain reasonable resemblance between original body and the assemblage. In order to overcome this drawback, isoparametric elements are used.

If the number of nodes used for defining the geometry is same as number of nodes used defining the displacements, then it is known as isoparametric element.



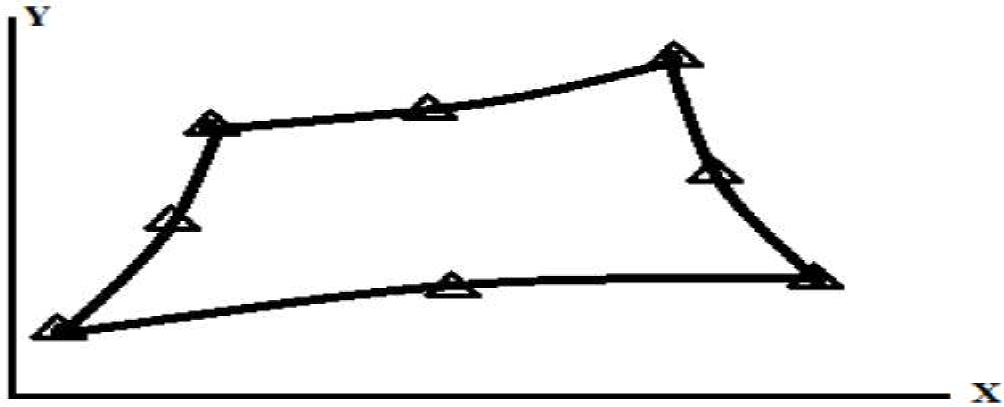
SUPERPARAMETRIC ELEMENT

If the number of nodes used for defining the geometry is more than number of nodes used for defining the displacements, then it is known as superparametric element.



SUBPARAMETRIC ELEMENT

If the number of nodes used for defining the geometry is less than number of nodes used for defining the displacements, then it is known as subparametric element.



● — **Nodes used for defining geometry**

△ — **Nodes used for defining displacements**

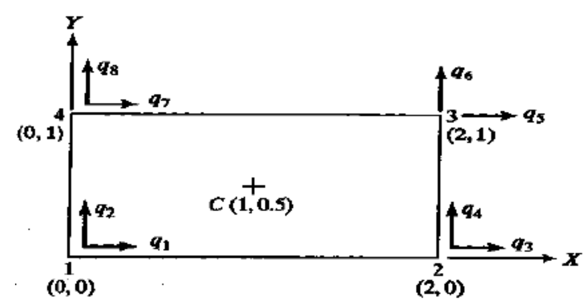
2. Consider a rectangular element as shown in Fig. . Assume plane stress condition, $E = 30 \times 10^6$ psi, $\nu = 0.3$, and $\mathbf{q} = [0, 0, 0.002, 0.003, 0.006, 0.0032, 0, 0]^T$ in. Evaluate $\mathbf{J}, \mathbf{B}$, and $\boldsymbol{\sigma}$ at $\xi = 0$ and $\eta = 0$.

Solution:

, we have

$$\mathbf{J} = \frac{1}{4}\begin{bmatrix} 2(1-\eta) + 2(1+\eta) & (1+\eta) - (1+\eta) \\ -2(1+\xi) + 2(1+\xi) & (1+\xi) + (1-\xi) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$$



FIGURE

For this rectangular element, we find that $\mathbf{J}$ is a constant matrix. Now, from Eqs.

$$\mathbf{A} = \frac{1}{1/2}\begin{bmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & \frac{1}{2} & 0 \end{bmatrix}$$

Evaluating G in Eq. 7.23 at $\xi = \eta = 0$ and using $\mathbf{B} = \mathbf{QG}$, we get

$$\mathbf{B}^0 = \begin{bmatrix} -\frac{1}{4} & 0 & \frac{1}{4} & 0 & \frac{1}{4} & 0 & -\frac{1}{4} & 0 \\ 0 & -\frac{1}{2} & 0 & -\frac{1}{2} & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{4} & -\frac{1}{2} & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & \frac{1}{2} & -\frac{1}{4} \end{bmatrix}$$

The stresses at $\xi = \eta = 0$ are now given by the product

$$\boldsymbol{\sigma}^0 = \mathbf{D}\mathbf{B}^0\mathbf{q}$$

For the given data, we have

$$\mathbf{D} = \frac{30 \times 10^6}{(1 - 0.09)} \begin{bmatrix} 1 & 0.3 & 0 \\ 0.03 & 1 & 0 \\ 0 & 0 & 0.35 \end{bmatrix}$$

Thus,

$$\boldsymbol{\sigma}^0 = [66\,920, 23\,080, 40\,960]^{\mathrm{T}} \text{ psi}$$

3. Evaluate

$$I = \int_{-1}^{1} \left[3e^x + x^2 + \frac{1}{(x + 2)} \right] dx$$

using one-point and two-point Gauss quadrature.

Solution For $n = 1$, we have $w_1 = 2$, $x_1 = 0$, and

$$I \approx 2f(0)$$
$$= 7.0$$

For $n = 2$, we find $w_1 = w_2 = 1$, $x_1 = -0.57735\ldots$, $x_2 = +0.57735\ldots$, and $I \approx 8.7857$. This may be compared with the exact solution

$$I_{\text{exact}} = 8.8165$$

4. Use Gaussian quadrature to obtain an exact value for the integral

$$I = \int_{-1}^{1} \int_{-1}^{1} (r^3 - 1)(s - 1)^2 \, dr \, ds$$

Solution

Considering first the integration with respect to r, we have a cubic order that requires two sampling points, which from Table 6.1 are given as $r_i = \pm 0.5773503$, and each of the corresponding weighting factors is unity. Similarly, for the integration with respect to s, the order is quadratic so the factors are the same. (In the following solution, we note, for simplicity of presentation, that the sampling points are numerically equal to $\sqrt{3}/3$.) Equation 6.106 is then, for this example,

$$I = \sum_{j=1}^{2} \sum_{i=1}^{2} W_j W_i f(r_i, s_j)$$

$$= \left[\left(\frac{\sqrt{3}}{3} \right)^3 - 1 \right] \left(\frac{\sqrt{3}}{3} - 1 \right)^2 + \left[\left(\frac{-\sqrt{3}}{3} \right)^3 - 1 \right] \left(\frac{\sqrt{3}}{3} - 1 \right)^2$$

$$+ \left[\left(\frac{\sqrt{3}}{3} \right)^3 - 1 \right] \left(\frac{-\sqrt{3}}{3} - 1 \right)^2 + \left[\left(\frac{-\sqrt{3}}{3} \right)^3 - 1 \right] \left(\frac{-\sqrt{3}}{3} - 1 \right)^2 = 5.33333333$$

UNIT VI

1.explain Consistent mass matrix and Lumped mass matrix.

CONSISTENT MASS MATRIX

1. The mass of each element is equally distributed at all the nodes of that Element.
2. Mass, being a scalar quantity, has same effect along the three translational degrees of freedom (u, v and w) and is not shared
3. Mass, being a scalar quantity, is not influenced by the local or global coordinate system. Hence, no transformation matrix is used for converting mass matrix from element (or local) coordinate system to structural (or global) coordinate system.

LUMPED MASS MATRIX

Total mass of the element is assumed equally distributed at all the nodes of the element in each of the translational degrees of freedom. Lumped mass is not used for rotational degrees of freedom. Off-diagonal elements of this matrix are all zero.

2. A composite wall consists of three materials, as shown in Fig. E10.1a. The outer temperature is $T_0 = 20°C$. Convection heat transfer takes place on the inner surface of the wall with $T_\infty = 800°C$ and $h = 25\ W/m^2 \cdot °C$. Determine the temperature distribution in the wall.

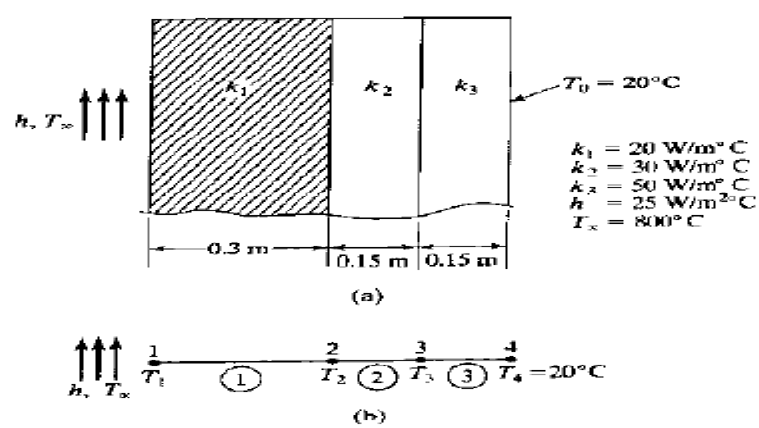


FIGURE E10.1

Solution A three-element finite element model of the wall is shown in Fig. E10.1b. The element conductivity matrices are

$$\mathbf{k}_T^{(1)} = \frac{20}{0.3}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \qquad \mathbf{k}_T^{(2)} = \frac{30}{0.15}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$\mathbf{k}_T^{(3)} = \frac{50}{0.15}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

The global $\mathbf{K} = \Sigma \mathbf{k}_T$ is obtained from these matrices as

$$\mathbf{K} = 66.7\begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 4 & -3 & 0 \\ 0 & -3 & 8 & -5 \\ 0 & 0 & -5 & 5 \end{bmatrix}$$

Now, since convection occurs at node 1, the constant $h = 25$ is added to the (1, 1) location of $\mathbf{K}$. This results in

$$\mathbf{K} = 66.7\begin{bmatrix} 1.375 & -1 & 0 & 0 \\ -1 & 4 & -3 & 0 \\ 0 & -3 & 8 & -5 \\ 0 & 0 & -5 & 5 \end{bmatrix}$$

Since no heat generation Q occurs in this problem, the heat rate vector $\mathbf{R}$ consists only of hT_∞ in the first row. That is,

$$\mathbf{R} = [25 \times 800, \quad 0, \quad 0, \quad 0]^T$$

The specified temperature boundary condition $T_4 = 20°C$, will now be handled by the penalty approach. We choose C based on

$$C = \max |\mathbf{K}_{ij}| \times 10^4$$
$$= 66.7 \times 8 \times 10^4$$

Now, C gets added to (4, 4) location of $\mathbf{K}$, while CT_4 is added to the fourth row of $\mathbf{R}$. The resulting equations are

$$66.7\begin{bmatrix} 1.375 & -1 & 0 & 0 \\ -1 & 4 & -3 & 0 \\ 0 & -3 & 8 & -5 \\ 0 & 0 & -5 & 80\,005 \end{bmatrix}\begin{Bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \end{Bmatrix} = \begin{Bmatrix} 25 \times 800 \\ 0 \\ 0 \\ 10\,672 \times 10^4 \end{Bmatrix}$$

The solution is

$$\mathbf{T} = [304.6, \quad 119.0, \quad 57.1, \quad 20.0]^T °C$$

3. A metallic fin, with thermal conductivity $k = 360$ W/m · °C, 0.1 cm thick, and 10 cm long, extends from a plane wall whose temperature is 235°C. Determine the temperature distribution and amount of heat transferred from the fin to the air at 20°C with $h = 9$ W/m² · °C. Take the width of fin to be 1 m.

Solution Assume that the tip of the fin is insulated. Using a three-element finite element model (Fig. E10.3) and assembling $\mathbf{K}_T, \mathbf{H}_T, \mathbf{R}_\infty$ as given previously, we find that Eq. 10.40 yields

$$\left[\frac{360}{3.33 \times 10^{-2}}\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} + \frac{9 \times 3.33 \times 10^{-2}}{3 \times 10^{-3}}\begin{bmatrix} 4 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 2 \end{bmatrix}\right]\begin{Bmatrix} T_2 \\ T_3 \\ T_4 \end{Bmatrix}$$
$$= \frac{9 \times 20 \times 3.33 \times 10^{-2}}{10^{-3}}\begin{Bmatrix} 2 \\ 2 \\ 1 \end{Bmatrix} - \begin{Bmatrix} -10711 \times 235 \\ 0 \\ 0 \end{Bmatrix}$$

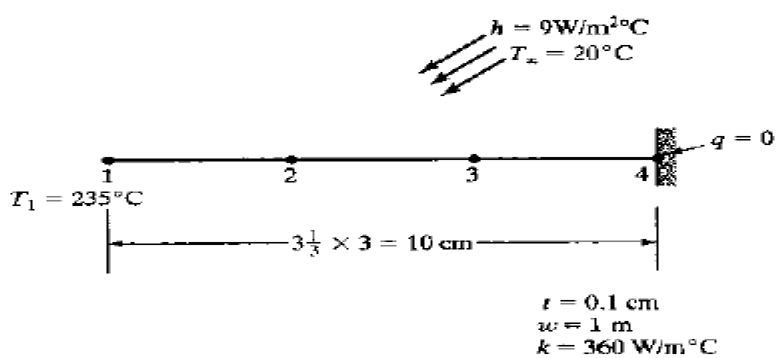


FIGURE E10.3

The solution is

$$[T_2, \quad T_3, \quad T_4] = [209.8, \quad 195.2, \quad 190.5]°C$$

The total heat loss in the fin can now be computed as

$$H = \sum_e H_e$$

The loss H_e in each element is

$$H_e = h(T_{av} - T_\infty)A_s$$

where $A_s = 2 \times (1 \times 0.0333)$ m², and T_{av} is the average temperature within the element. We obtain

$$H_{loss} = 334.3 \text{ W/m}$$

4. Determine the eigenvalues and eigenvectors for the stepped bar shown in Fig.

Solution Gathering the stiffness and mass values corresponding to the degrees of freedom Q_2 and Q_3, we get the eigenvalue problem

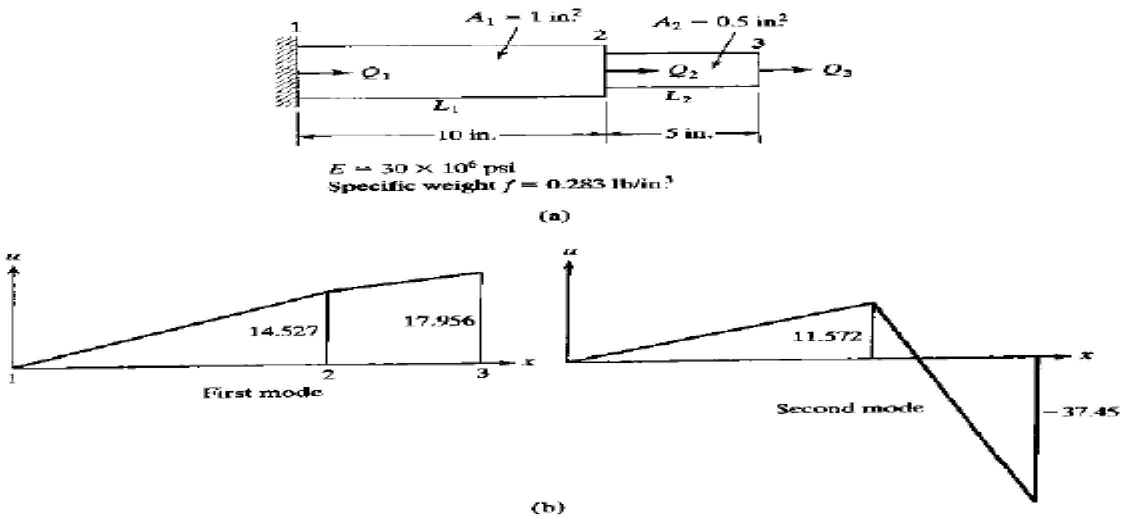
$$E\begin{bmatrix} \left(\frac{A_1}{L_1}+\frac{A_2}{L_2}\right) & -\frac{A_2}{L_2} \\ -\frac{A_2}{L_2} & \frac{A_2}{L_2} \end{bmatrix}\begin{Bmatrix} U_2 \\ U_3 \end{Bmatrix} = \lambda\frac{\rho}{6}\begin{bmatrix} 2(A_1L_1 + A_2L_2) & A_2L_2 \\ A_2L_2 & 2A_2L_2 \end{bmatrix}\begin{Bmatrix} U_2 \\ U_3 \end{Bmatrix}$$

We note here that the density is

$$\rho = \frac{f}{g} = \frac{0.283}{32.2 \times 12} = 7.324 \times 10^{-4}\ \text{lbs}^2/\text{in.}^4$$

Substituting the values, we get

$$30 \times 10^6\begin{bmatrix} 0.2 & -0.1 \\ -0.1 & 0.1 \end{bmatrix}\begin{Bmatrix} U_2 \\ U_3 \end{Bmatrix} = \lambda 1.22 \times 10^{-4}\begin{bmatrix} 25 & 2.5 \\ 2.5 & 5 \end{bmatrix}\begin{Bmatrix} U_2 \\ U_3 \end{Bmatrix}$$



FIGURE

The characteristic equation is

$$\det\begin{bmatrix} (6 \times 10^6 - 30.5 \times 10^{-4}\lambda) & (-3 \times 10^6 - 3.05 \times 10^{-4}\lambda) \\ (-3 \times 10^6 - 3.05 \times 10^{-4}\lambda) & (3 \times 10^6 - 6.1 \times 10^{-4}\lambda) \end{bmatrix} = 0$$

which simplifies to

$$1.77 \times 10^{-6}\lambda^2 - 1.465 \times 10^4\lambda + 9 \times 10^{12} = 0$$

The eigenvalues are

$$\lambda_1 = 6.684 \times 10^8$$

$$\lambda_2 = 7.61 \times 10^9$$

Note that $\lambda = \omega^2$, where ω is the circular frequency given by $2\pi f$ and f = frequency in hertz (cycles/s).

These frequencies are

$$f_1 = 4115\ \text{Hz}$$

$$f_2 = 13\,884\ \text{Hz}$$

The two previous equations are not independent, since the determinant of the matrix is zero. This gives

$$3.96U_2 = 3.204U_3$$

Thus,

$$\mathbf{U}_1^{\mathrm{T}} = [U_2, 1.236U_2]$$

For normalization, we set

$$\mathbf{U}_1^{\mathrm{T}}\mathbf{M}\mathbf{U}_1 = 1$$

On substituting for $\mathbf{U}_1$, we get

$$\mathbf{U}_1^{\mathrm{T}} = [14.527 \quad 17.956]$$

The eigenvector corresponding to the second eigenvalue is similarly found to be

$$\mathbf{U}_2^{\mathrm{T}} = [11.572 \quad -37.45]$$

The mode shapes are shown in Fig.